— Lesson № 1 —

The Complex Number System $\mathbb{C}$ and Arithmetic Operations on $\mathbb{C} \times \mathbb{C}$

«The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and not being».

— *Gottfried Wilhelm Leibniz (1646–1716)*

— Overview. Complex numbers are used in many scientific fields, including engineering, electromagnetism, quantum physics and applied mathematics, such as chaos theory. Therefore, knowledge of complex numbers is clearly absolutely essential for further studies in so many engineering disciplines. Lesson № 1 introduces the definition and the basic properties of the *complex number system* $\mathbb{C}$. This is followed by a presentation of the fundamental *arithmetic operations on* $\mathbb{C} \times \mathbb{C}$.

— Key Words and Phrases. *Complex numbers, real and imaginary parts, geometric representation of a complex number, complex number system, arithmetic operations on complex numbers, modulus, conjugate, Argand diagram, properties of complex numbers*

— Learning Outcomes. After doing Assignment 1.1, the student will be able to:

- I. explain how quadratic equations in $\left\{\alpha z^2 + \beta z + \gamma = 0 : \ (\alpha, \beta, \gamma) \in \mathbb{R}^3\right\}$ lead to the field $\mathbb{C} = \left(\left\{z = x + yi : \ (x, y) \in \mathbb{R}^2\right\}; +, \times\right)$ of complex numbers and how to plot complex numbers in $\mathbb{C}$ on a Argand diagram in $\mathbb{R} \times \mathbb{R}$,
- II. perform basic arithmetic operations on $\mathbb{C} \times \mathbb{C}$ and interpret the results in Cartesian, polar and exponential forms,
- III. explain Euler's formula $\exp(i\theta) = \cos(\theta) + i \sin(\theta)$ and the exponential form $z = r \exp(i\theta)$ of a complex number $z = x + yi$,
- IV. use de Moivre's formula $\Big(r\big(\cos(\theta) + i \sin(\theta)\big)\Big)^n = r^n\big(\cos(n\theta) + i \sin(n\theta)\big)$.

— § 1.1. Introduction

The most primitive type of number system is the *natural number system* $\mathbb{N} = (\mathcal{N};+,\times)$, where $\mathcal{N} = \{1,2,3,\ldots\}$. Arithmetic, the *science of numbers*, is based on the fact that numbers can be added and multiplied by the addition and multiplication laws $\begin{array}{ll} +,\ \times : \ \mathcal{N}\times\mathcal{N} & \longrightarrow \mathcal{N} \\ \qquad\quad (\alpha,\beta) & \longmapsto \alpha+\beta,\ \alpha\times\beta, \end{array}$ subject to certain rules. It is the existence of these two laws of composition and their mutual relation that we shall regard as the typical feature of all numbers and that will serve us as a guide for introducing new number systems for various purposes.

The attempt to make subtraction under the law $\begin{array}{ll} - : \ \mathcal{N}\times\mathcal{N} & \longrightarrow \mathcal{Z} \\ \qquad (\alpha,\beta) & \longmapsto \alpha-\beta \end{array}$ always possible, that is to solve the equation $\beta + x = \alpha$ for x when $(\alpha,\beta)\in\mathcal{N}\times\mathcal{N}$ is given, leads to the introduction of the *negative numbers* and *zero* $\ldots$, -3, -2, -1, 0. We now have the *integer number system* $\mathbb{Z} = (\mathcal{Z};+,\times)$, where $\mathcal{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. Next, when we wish to carry out division under the law $\begin{array}{ll} \div : \ \mathcal{Z}\times\mathcal{Z}^{*} & \longrightarrow \mathcal{Q} \\ \qquad (\alpha,\beta) & \longmapsto \alpha\div\beta, \end{array}$ we have to solve equations of the form $\beta x = \alpha$, where $(\alpha,\beta)\in\mathcal{Z}\times\mathcal{Z}^{*}$ is given. Thus, to make the solution possible in all cases it is necessary to introduce the *rational number system* $\mathbb{Q} = (\mathcal{Q};+,\times)$, where $\mathcal{Q} = \left\{\gamma = \dfrac{\alpha}{\beta} : \ (\alpha,\beta)\in\mathcal{Z}\times\mathcal{Z}^{*}\right\}$. When this stage has been reached, the four laws of arithmetic, that is addition, subtraction, multiplication and division

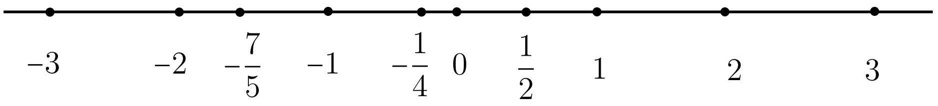
$$\begin{array}{ll} +,\ -,\ \times,\ \div : \ \mathcal{Q}\times\mathcal{Q} & \longrightarrow \mathcal{Q} \\ \qquad\qquad (\alpha,\beta) & \longmapsto \alpha+\beta,\ \alpha-\beta,\ \alpha\times\beta,\ \alpha\div\beta \end{array} \tag{1.1}$$

apply without restriction, always excepting division by zero. These basic operations are governed by the following general rules, which are of fundamental importance in mathematics.

- Commutative Law of +: $\big(\forall\,(\alpha,\beta)\in\mathcal{Q}^2\big)\big[\alpha+\beta=\beta+\alpha\big]$
- Associative Law of +: $\big(\forall\,(\alpha,\beta,\gamma)\in\mathcal{Q}^3\big)\big[(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)\big]$
- Law of −: $\big(\forall\,(\alpha,\beta)\in\mathcal{Q}\times\mathcal{Q}\big)\big(\exists! x\in\mathcal{Q}\big)\big[\beta+x=\alpha \ \longrightarrow\ x=\alpha-\beta\big]$
- Commutative Law of ×: $\big(\forall\,(\alpha,\beta)\in\mathcal{Q}^2\big)\big[\alpha\times\beta=\beta\times\alpha\big]$
- Associative Law of ×: $\big(\forall\,(\alpha,\beta,\gamma)\in\mathcal{Q}^3\big)\big[(\alpha\times\beta)\times\gamma=\alpha\times(\beta\times\gamma)\big]$
- Law of ÷: $\big(\forall\,(\alpha,\beta)\in\mathcal{Q}\times\mathcal{Q}^{*}\big)\big(\exists! x\in\mathcal{Q}\big)\big[\beta x=\alpha \ \longrightarrow\ x=\alpha\div\beta\big]$

– Distributive Law: $\left(\forall\,(\alpha,\beta,\gamma)\in\mathcal{Q}^3\right)\left[(\alpha+\beta)\times\gamma=(\alpha\times\gamma)+(\beta\times\gamma)\right]$

The rational number system $\mathbb{Q}=(\mathcal{Q};+,\times)$ is more adequate for dealing with the more elementary questions of arithmetic, but its deficiency becomes apparent when we consider such problems as extracting square roots. It can be shown that there exists no $\gamma\in\mathbb{Q}$ such that $\gamma=\sqrt{2}$. That is, $\sqrt{2}$ cannot be expressed in the form $\dfrac{\alpha}{\beta}$, where $(\alpha,\beta)\in\mathbb{Z}\times\mathbb{Z}^*$, such that $\alpha^2=2\times\beta^2$. Again, when we pass from Algebra to Analysis, where limits of sequences play a fundamental part, we find that the limit of a sequence of rational numbers $\langle\gamma_k\in\mathbb{Q}\rangle_{k\in I_\infty^*}$ is not necessarily a rational number. The situation may be described by using a single coordinate axis on which in the firts place we mark all the integers in a certain scale, as in Fig. 1.



— Figure 1. Geometrical illustration of *inexact* and *exact differentials.*

Then, we imagine all the rational numbers inserted, $-\frac{7}{5}$, $-\frac{1}{4}$, $\frac{1}{2}$, ... But even when this has been done, there will be many points on the line against which no number has been entered. For instance when we lay down a segment of length $\sqrt{2}$ by placing one end at 0, the other end-point falls on a point of the scale which has as yet no number attached to it. On the other hand, we intuitively accept the fact that every segment ought to have a length which is measured by some *number*. In other words, we consider the following postulate.

— Postulate 1.1. *Every point on the axis possesses a coordinate which is a definite number, positive if the point is on the right of* 0 *and negative if it is on the left of* 0. *This number need not be a rational number.*

The set of numbers $\mathcal{R}=\left\{\alpha:\ \left(\alpha\in\mathcal{Q}\right)\vee\left(\alpha\in\complement\left(\mathcal{Q}\right)\right)\right\}$ which in this way fill the whole line, is called the *set of real numbers*, and $\mathbb{R}=(\mathcal{R};+,\times)$ is called the *real number system.* Therefore, $\mathcal{R}$ comprises the set of rational numbers $\mathcal{Q}$ and the set of *irrational numbers* $\complement\left(\mathcal{Q}\right)=\left\{\alpha:\ \alpha\notin\mathcal{Q}\right\}=\left\{\sqrt{2},\ \mathrm{e},\ \pi,\log\left(2\right),\ \ldots\right\}$.

— Remark 1.2. The word *irrational* means that the number is not the *ratio* of two integers and has nothing to do with the idea that something irrational is beyond the realm of reason.

From the way in which real numbers are depicted on a line it is clear that there exists an *order relation* among them, that is for any $(\alpha,\beta)\in\mathbb{R}^2$, either $\alpha<\beta$ or $\alpha=\beta$ or $\beta<\alpha$. This is indeed an important property when we wish to use numbers for measuring.

For a long time it was held that arithmetic had reached saturation with the introduction of the real number system $\mathbb{R} = (\mathcal{R};+,\times)$. Indeed, there was no obvious geometrical or technical problem that called for the creation of new numbers. Yet, one of the simplest algebraical questions remains in an unsatisfactory state when only the system $\mathbb{R} = (\mathcal{R};+,\times)$ are available. For we should then be forced to admit that some quadratic equations in $\left\{\alpha\zeta^2+\beta\zeta+\gamma=0 : (\alpha,\beta,\gamma)\in\mathbb{R}^3\right\}$ have solutions whilst others have none. On the other hand, it is easy to see that all quadratic equations would have solutions if only the special quadratic equation

$$\zeta^2+1=0 \tag{1.2}$$

could be solved. For, this would assign a meaning to $\sqrt{-1}$ and hence to $\sqrt{-\eta}$, where $\eta\in\mathbb{R}_+^*$. Now, it is clear that Eq. (1.2) cannot have a real solution, since if $\zeta\in\mathbb{R}$, then $\zeta^2\notin\mathbb{R}_-$ and the equation $\zeta^2=-1$ cannot therefore hold. In order to make Eq. (1.2) soluble, a new type of number should be introduced, for which the statement $\zeta^2\in\mathbb{R}_+$ certainly does not hold.

— § 1.2. The Complex Number System $\mathbb{C}$

1.2.1. The Structure $\mathbb{C}$. Following the discussion under § 1.1, in the formal solution of the equation $\alpha\zeta^2+\beta\zeta+\gamma=0$, where $\beta^2<4\alpha\gamma$ and $(\alpha,\beta,\gamma)\in\mathbb{R}^3$, numbers of the form $\xi+i\eta$, where $(\xi,\eta)\in\mathbb{R}^2$, arise. Thus, it is customary to write $\sqrt{-1}=i$ and to treat i as a literal quantity subject to the relation $i^2=-1$.

— Definition 1.3 (*Imaginary Unit*). A symbol i such that

$$i^2=-1 \tag{1.3}$$

is called an «imaginary unit».

Then, an imaginary is not a quantity, and the treatment of imaginaries is pure arbitrary and conventional. The following lemma the results we can get for powers of $i=\sqrt{-1}$, that is, for repetitions of the operation indicated by the symbol.

— Lemma 1.4. *If i be the imaginary unit satisfying the equation $i^2=-1$, then*

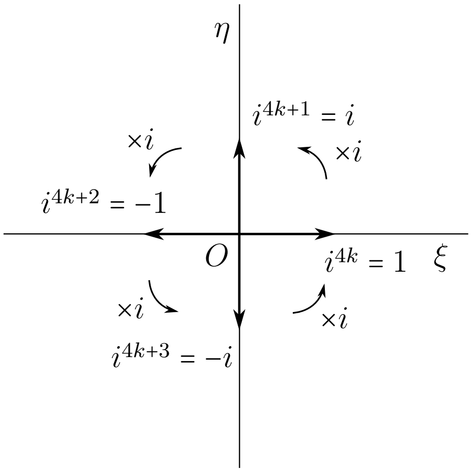
$$\left(\forall k\in\mathbb{Z}_+^0\right)\left[\left(i^{4k}=1\right)\wedge\left(i^{4k+1}=i\right)\wedge\left(i^{4k+2}=-1\right)\wedge\left(i^{4k+3}=-i\right)\right] \tag{1.4}$$

— Proof. Let i be the imaginary unit satisfying the equation $i^2=-1$. Then,

$$i^{4k}=\left(i^4\right)^k=\left(i^2\times i^2\right)^k=\left((-1)\times(-1)\right)^k=1^k=1$$

for any $k\in\mathbb{Z}_+^0$. Thus, $i^{4k}=1$. Since $i^{4k}=1$, then $i^{4k+1}=i^{4k}\times i=1\times i=i$. Hence, $i^{4k+1}=i$. Since $i^{4k+1}=i$, then $i^{4k+2}=i^{4k}\times i^2=1\times(-1)=-1$. Thus, $i^{4k+2}=-1$. Since $i^{4k+2}=-1$, then $i^{4k+3}=i^{4k+2}\times i=-1\times i=-i$. Hence, $i^{4k+3}=-i$. The proof of the lemma is, therefore, complete. Q.e.d.

The geometrical interpretation of LEM. 1.4 in a $(\xi, \eta$-plane in $\mathbb{R}^2$ is that the effect of multiplying a line segment by the imaginary unit i is to turn through a right angle. In FIG. 2 is evidenced this fact.



— FIGURE 2. Geometrical interpretation of multiplication by i in a (ξ, η)-plane in $\mathbb{R}^2$.

— ILLUSTRATIVE EXAMPLE 1.5. Let i be the imaginary unit satisfying the equation $i^2 = -1$. For each $(k, \gamma) \in \{k : k \in I_4^*\} \times \mathbb{Z}_+$, calculate $i^{p_k(\gamma)}$, where:

- I. $p_1(\gamma) = 16\gamma^2 + 16\gamma + 3$
- II. $p_2(\gamma) = 16\gamma^2 + 20\gamma + 6$
- III. $p_3(\gamma) = 16\gamma^2 + 8\gamma$
- IV. $p_4(\gamma) = 16\gamma^2 + 12\gamma + 2$

– SOLUTION. Factoring the polynomials $p_1(\gamma), \ldots, p_4(\gamma)$ in $\gamma \in \mathbb{Z}_+$ gives:

- I. $p_1(\gamma) = (4\gamma + 3)(4\gamma + 1)$
- II. $p_2(\gamma) = (4\gamma + 3)(4\gamma + 2)$
- III. $p_3(\gamma) = \gamma(4\gamma + 2)$
- IV. $p_4(\gamma) = (4\gamma + 2)(4\gamma + 1)$

respectively. Then,

- I. $i^{p_1(\gamma)} = i^{4\gamma+3} \times i^{4\gamma+1}$
- II. $i^{p_2(\gamma)} = i^{4\gamma+3} \times i^{4\gamma+2}$
- III. $i^{p_3(\gamma)} = i^{4\gamma} \times i^{4\gamma+2}$
- IV. $i^{p_4(\gamma)} = i^{4\gamma+2} \times i^{4\gamma+1}$

respectively. But,

- I. $i^{4\gamma+3} \times i^{4\gamma+1} = (-i) \times i = 1$
- II. $i^{4\gamma+3} \times i^{4\gamma+2} = (-i) \times (-1) = i$
- III. $i^{4\gamma} \times i^{4\gamma+2} = 1 \times (-1) = -1$
- IV. $i^{4\gamma+2} \times i^{4\gamma+1} = (-1) \times i = -i$

Thus, $i^{p_1(\gamma)} = 1$, $i^{p_2(\gamma)} = i$, $i^{p_3(\gamma)} = -1$, and $i^{p_4(\gamma)} = -i$.

— THEOREM 1.6. *Suppose* $p_n(i) = \lambda_0 + \sum_{k\in I_n^*}\lambda_k i^k$ *be a polynomial in* i, *where* $\{\lambda_k :\ k\in I_n^0\}\subset\mathbb{R}$, *then there exists* $(\alpha,\beta)\in\mathbb{R}^2$ *such that* $p_n(i) = \alpha + i\beta$.

— PROOF. Suppose $p_n(i) = \lambda_0 + \sum_{k\in I_n^*}\lambda_k i^k$ be a polynomial in i, where $\{\lambda_k :\ k\in I_n^0\}\subset\mathbb{R}$. Then,

$$\begin{aligned} p_n(i) &= \lambda_0 + \sum_{k\in I_n^*}\lambda_k i^k \\ &= \lambda_0 + \sum_{k\in I_n^*}\lambda_{4k} i^{4k} + \sum_{k\in I_n^*}\lambda_{4k+1} i^{4k+1} + \sum_{k\in I_n^*}\lambda_{4k+2} i^{4k+2} + \sum_{k\in I_n^*}\lambda_{4k+3} i^{4k+3} \\ &= \lambda_0 + \sum_{k\in I_n^*}(\lambda_{4k+1}-\lambda_{4k+3}) + i\left(\sum_{k\in I_n^*}(\lambda_{4k}-\lambda_{4k+2})\right) \end{aligned}$$

Thus, there exists $(\alpha,\beta) = \left(\lambda_0 + \sum_{k\in I_n^*}(\lambda_{4k+1}-\lambda_{4k+3}),\ \sum_{k\in I_n^*}(\lambda_{4k}-\lambda_{4k+2})\right)\in\mathbb{R}^2$ such that $p_n(i) = \alpha + i\beta$. The proof of the theorem is, therefore, complete. Q.E.D.

In regard to THM. 1.6, it follows that algebraic operations with respect to the imaginary unit i does suggest numbers of the type $\xi + i\eta$, where $(\xi,\eta)\in\mathbb{R}^2$. Thus, $\xi + i\eta$ may therefore for our purposes be taken as the most general type of a complex number, and the definition follows.

— DEFINITION 1.7 (*Complex Number*). A symbol of the form

$$\zeta = \xi + i\eta \tag{1.5}$$

where $(\xi,\eta)\in\mathbb{R}^2$, is called a «complex number in algebraic form», and the set $\mathcal{C} = \{\zeta = \xi + i\eta :\ (x,y)\in\mathbb{R}^2\}$ is called the «set of complex numbers».

The set $\mathcal{C}$ together with the laws $\begin{array}{ccl} +,\ \times : & \mathcal{C}\times\mathcal{C} & \longrightarrow \mathcal{C} \\ & (\zeta_1,\zeta_2) & \longmapsto \zeta_1+\zeta_2,\ \zeta_1\times\zeta_2 \end{array}$ of addition and multiplication forms the system $\mathbb{C} = (\mathcal{C};+,\times)$, called the *complex number system*.

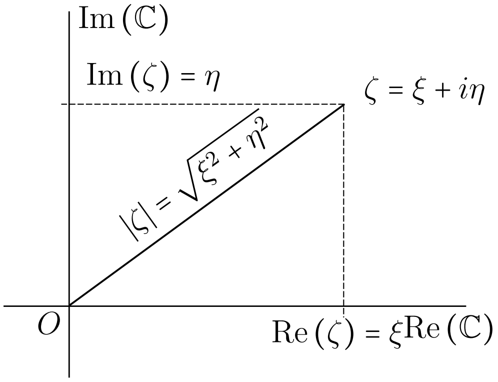
— DEFINITION 1.8 (*Complex Number System*). The system $\mathbb{C} = (\mathcal{C};+,\times)$ formed by the set $\mathcal{C} = \{\zeta = \xi + i\eta :\ (\xi,\eta)\in\mathbb{R}^2\}$, and the binary laws

$$\begin{array}{ccl} +,\ \times : & \mathcal{C}\times\mathcal{C} & \longrightarrow \mathcal{C} \\ & (\xi_1 + i\eta_1,\ \xi_2 + i\eta_2) & \longmapsto (\xi_1+\xi_2) + i(\eta_1+\eta_2), \\ & & \quad (\xi_1\xi_2 - \eta_1\eta_2) + i(\xi_1\eta_2 + \xi_2\eta_1) \end{array} \tag{1.6}$$

is called the «complex number system».

Evidently, if ζ_1, $\zeta_2 \in \mathbb{C}$ such that $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$, where (ξ_1, η_1), $(\xi_2, \eta_2) \in \mathbb{R}^2$, then $\zeta_1 + \zeta_2 = (\xi_1 + \xi_2) + i(\eta_1 + \eta_2)$ and $\zeta_1 \times \zeta_2 = (\xi_1\xi_2 - \eta_1\eta_2) + i(\xi_1\eta_2 + \xi_2\eta_1)$ are the *sum* and *product* of ζ_1, ζ_2, respectively.

While real numbers may be arranged geometrically on a straight line according to magnitude with some arbitrary point taken as the origin, in order to represent a complex number, a plane is needed. This is customarily done by labeling the axis of abscissas the real axis (Re) and the axis of ordinates the imaginary axis (Im). Thus, the complex number $\zeta \in \mathbb{C}$ such that $\zeta = \xi + i\eta$ is represented on the Cartesian (ξ, η)-plane in $\mathbb{R}^2$ as the point with coordinates $(\xi, \eta) \in \mathbb{R}^2$, as in FIG. 3.



— FIGURE 3. Geometrical Representation of a Complex Number in a (ξ, η)-plane in $\mathbb{R}^2$.

— ILLUSTRATIVE EXAMPLE 1.9. Let γ_1, $\gamma_2 \in \mathbb{C}$ be any two complex numbers such that $\gamma_1 = 2\alpha + i3\beta$ and $\gamma_2 = 3\alpha + i5\beta$, where $(\alpha, \beta) \in \mathbb{R}^2$. Calculate the sum $\gamma_1 + \gamma_2$ and the product $\gamma_1 \times \gamma_2$ of the complex numbers γ_1, γ_2.

– SOLUTION. Since

$$\gamma_1 + \gamma_2 = (2\alpha + 3\alpha) + i(3\beta + 5\beta) = 5\alpha + i8\beta$$

then $\gamma_1 + \gamma_2 = 5\alpha + i8\beta$ is the sum of ζ_1, ζ_2. Since

$$\begin{aligned} \gamma_1 \times \gamma_2 &= \big((2\alpha)(3\alpha) - (3\beta)(5\beta)\big) + i\big((2\alpha)(5\beta) + (3\alpha)(5\beta)\big) \\ &= \left(6\alpha^2 - 15\beta^2\right) + i25\alpha\beta \end{aligned}$$

then $\gamma_1 \times \gamma_2 = \left(6\alpha^2 - 15\beta^2\right) + i25\alpha\beta$ is the product of ζ_1, ζ_2.

For any complex number $\zeta \in \mathbb{C}$ such that $\zeta = \xi + i\eta$, it is convenient to call the real ξ and the imaginary η the *real* and *imaginary* parts of ζ, respectively.

— DEFINITION 1.10 (*Real and Imaginary Parts*). Let $\zeta \in \mathbb{C}$ be a complex number such that $\zeta = \xi + i\eta$, where $(\xi, \eta) \in \mathbb{R}^2$. Then:

– I. $\mathrm{Re}(\zeta) = \xi$ is called the «real part» of ζ,

– II. $\mathrm{Im}(\zeta) = \eta$ is called the «imaginary part» of ζ.

Clearly, $\zeta = \mathrm{Re}(\zeta) + i\,\mathrm{Im}(\zeta)$ for any $\zeta \in \mathbb{C}$. Thus, a complex number $\zeta \in \mathbb{C}$ which is such that $\mathrm{Im}(\zeta) = 0$ is called a (pure) *real number* while a complex number $\zeta \in \mathbb{C}$ which is such that $\mathrm{Re}(\zeta) = 0$ is called a (pure) *imaginary number*.

— DEFINITION 1.11 (*Equality*). Let ζ_1, $\zeta_2 \in \mathbb{C}$ be any two complex numbers. Then, ζ_1, ζ_2 are said to be «equal» if and only if their real and imaginary parts are equal:

$$\zeta_1 = \zeta_2 \;\Leftrightarrow \begin{cases} \mathrm{Re}(\zeta_1) = \mathrm{Re}(\zeta_2) \\ \mathrm{Im}(\zeta_1) = \mathrm{Im}(\zeta_2) \end{cases} \tag{1.7}$$

— DEFINITION 1.12 (*Modulus*). Let $\zeta \in \mathbb{C}$ be any complex number such that $\zeta = \xi + i\eta$, where $(\xi, \eta) \in \mathbb{R}^2$. Then, the positive square root of $\xi^2 + \eta^2$ is called the «modulus» of ζ:

$$|\zeta| = \sqrt{\xi^2 + \eta^2} = \left(\left(\mathrm{Re}(\zeta)\right)^2 + \left(\mathrm{Im}(\zeta)\right)^2\right)^{1/2} \tag{1.8}$$

Thus, the modulus of a complex number in $\mathbb{C}$ is the natural extension of the concept of *absolute value* of a real number in $\mathbb{R}$. Geometrically, the modulus is the *distance* from the origin O to the point $(\xi, \eta) \in \mathbb{R}^2$ in the (ξ, η)-plane in $\mathbb{R}^2$.

— ILLUSTRATIVE EXAMPLE 1.13. Let γ_1, $\gamma_2 \in \mathbb{C}$ be any two complex numbers such that $\gamma_1 = 2\alpha + i3\beta$ and $\gamma_2 = 3\alpha + i5\beta$, where $(\alpha, \beta) \in \mathbb{R}^2$. Calculate the moduli $|\gamma_1|$, $|\gamma_2|$, $|\gamma_1 + \gamma_2|$, and $|\gamma_1 \times \gamma_2|$.

– SOLUTION. Since $\gamma_1 = 2\alpha + i3\beta$ and $\gamma_2 = 3\alpha + i5\beta$, then $|\gamma_1| = \sqrt{4\alpha^2 + 9\beta^2}$, $|\gamma_2| = \sqrt{9\alpha^2 + 25\beta^2}$ are the moduli of γ_1, γ_2, respectively. Since $\gamma_1 + \gamma_2 = 5\alpha + i8\beta$ and $\gamma_1 \times \gamma_2 = \left(6\alpha^2 - 15\beta^2\right) + i25\alpha\beta$, then

$$\begin{cases} |\gamma_1 + \gamma_2| = \sqrt{25\alpha^2 + 64\beta^2} \\ |\gamma_1 \times \gamma_2| = \sqrt{\left(6\alpha^2 - 15\beta^2\right)^2 + (25\alpha\beta)^2} \end{cases}$$

are the moduli of the sum $\gamma_1 + \gamma_2$ and product $\gamma_1 \times \gamma_2$ of γ_1, γ_2, respectively.

— PROPOSITION 1.14. *If* ζ_1, $\zeta_2 \in \mathbb{C}$ *be any two complex numbers such that* $\zeta_1 \times \zeta_2 = 0$*, then either* $\zeta_1 = 0$ *or* $\zeta_2 = 0$*.*

— PROOF. Let ζ_1, $\zeta_2 \in \mathbb{C}$ be any two complex numbers such that $\zeta_1 \times \zeta_2 = 0$. Suppose $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$, where $\xi_1, \ldots, \eta_2 \in \mathbb{R}$. Then, $\zeta_1 \times \zeta_2 = (\xi_1\xi_2 - \eta_1\eta_2) + i(\xi_1\eta_2 + \xi_2\eta_1)$. Since $\zeta_1 \times \zeta_2 = 0$, it follows that $\xi_1\xi_2 - \eta_1\eta_2 = 0$ and

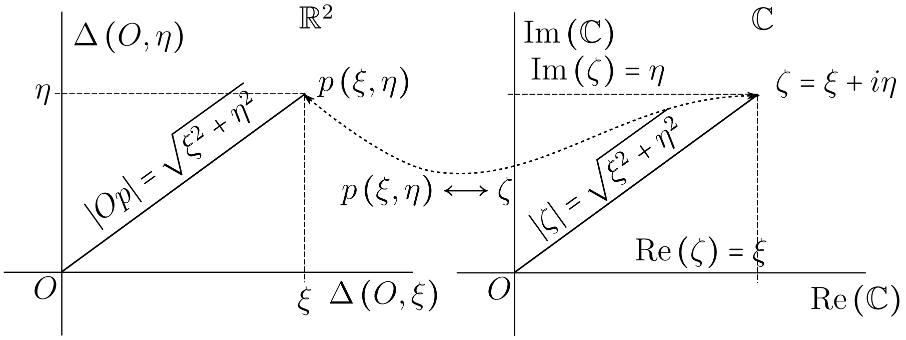
$\xi_1\eta_2 + \xi_2\eta_1 = 0$, implying either $(\xi_1, \eta_1) = (0,0)$ or $(\xi_2, \eta_2) = (0,0)$ holds true. Thus, the equation $\zeta_1 \times \zeta_2 = 0$ implies either $\zeta_1 = 0$ or $\zeta_2 = 0$. The proof of the proposition is, therefore, complete. Q.E.D.

1.2.2. THE ARGAND DIAGRAM OF $\mathbb{C}$. In § 1.2.1, we have seen that complex numbers in $\mathbb{C}$ may be represented in a geometrical diagram by taking Cartesian axes $\Delta(O,\xi)$, $\Delta(O,\eta)$ in a (ξ,η)-plane in $\mathbb{R}^2$. Then a point $p(\xi,\eta)$ whose coordinates referred to the axes $\Delta(O,\xi)$, $\Delta(O,\eta)$ are ξ, η, respectively, may be regarded as representing the complex number $\zeta \in \mathbb{C}$ such that $\zeta = \xi + i\eta$. In this way, to every point of the (ξ,η)-plane there corresponds some one complex number, and conversely, to every possible complex number there corresponds one, and only one, point of that plane.: $p(\xi,\eta) \longleftrightarrow \zeta = \xi + i\eta$.

— DEFINITION 1.15 (*Representative Point*). Let $p(\xi,\eta) \in \mathbb{R}^2$ on a (ξ,η)-plane in $\mathbb{R}^2$ be a point which corresponds to a unique complex number $\zeta \in \mathbb{C}$ such that $\zeta = \xi + i\eta$ in the complex number system $\mathbb{C}$. Then, $p(\xi,\eta)$ is called the «representative point» of ζ, and ζ is called the «affix» of $p(\xi,\eta)$

$$p(\xi,\eta) \longleftrightarrow \zeta = \xi + i\eta \tag{1.9}$$

The representation of complex numbers of $\mathbb{C}$ thus afforded is often called the *Argand diagram*, as in FIG. 4. Moreover, since $|Op(\xi,\eta)| = \sqrt{\xi^2+\eta^2}$ and $p(\xi,\eta) \longleftrightarrow \zeta = \xi + i\eta$, then $|\zeta| = \sqrt{\xi^2+\eta^2}$ is also often called the *modulus* of $\zeta \in \mathbb{C}$.

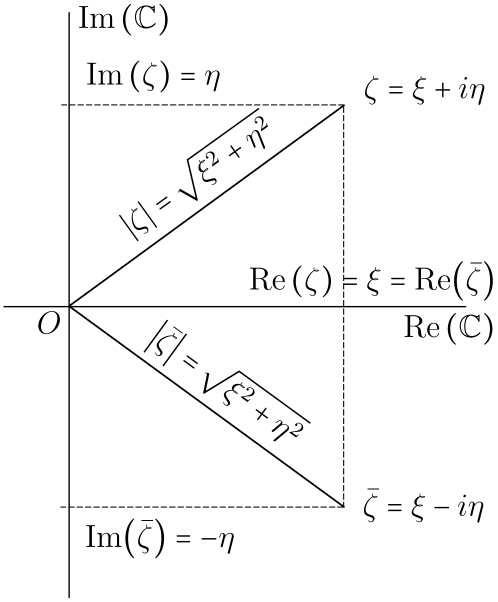


— FIGURE 4. Geometrical illustration of the representative point $p(\xi,\eta) \in \mathbb{R}^2$ of a complex number $\zeta(\xi,\eta) \in \mathbb{C}$.

— DEFINITION 1.16 (*Conjugate*). Let $\mathbb{C} = (\mathcal{C}; +, \times)$ be the complex number system. Then, the transformation

$$\begin{aligned} \mathrm{conj}: \quad & \mathcal{C} \longrightarrow \mathcal{C} \\ \zeta = \xi + i\eta \; & \longmapsto \mathrm{conj}(\zeta) = \xi - i\eta \end{aligned} \tag{1.10}$$

which replaces a complex number $\zeta = \xi + i\eta$ by $\mathrm{conj}(\zeta) = \xi - i\eta$ is called «complex conjugation», and $\bar{\zeta} = \xi - i\eta$ is called the «conjugate» of ζ.



— FIGURE 5. Geometrical Representation of the Conjugate of a Complex Number in a (ξ, η)-plane in $\mathbb{R}^2$.

— § 1.3. ARITHMETIC OPERATIONS ON $\mathbb{C} \times \mathbb{C}$

The definitions of addition and multiplication of complex numbers in $\mathbb{C}$ have been so framed that we may perform the ordinary operations of algebra with complex numbers in exactly the same way as with real numbers in $\mathbb{R}$, treating the imaginary unit $i = \sqrt{-1}$ as a number and using $i^2 = -1$ wherever it occurs. It readily follows that the *commutative*, *associative*, and *distributive laws* hold in $\mathbb{C}$.

— THEOREM 1.17. *If* ζ_1, ζ_2, $\zeta_3 \in \mathbb{C}$ *be any three complex numbers, then:*

– *I.* $\zeta_1 + \zeta_2 = \zeta_2 + \zeta_1$ – *II.* $(\zeta_1 + \zeta_2) + \zeta_3 = \zeta_1 + (\zeta_2 + \zeta_3)$

– *III.* $(\zeta_1 + \zeta_2) \times \zeta_3 = (\zeta_1 \times \zeta_3) + (\zeta_2 \times \zeta_3)$ – *IV.* $(\zeta_1 \times \zeta_2) \times \zeta_3 = \zeta_1 \times (\zeta_2 \times \zeta_3)$

– *V.* $\zeta_1 \times (\zeta_2 + \zeta_3) = (\zeta_1 \times \zeta_2) + (\zeta_1 \times \zeta_3)$ – *VI.* $\zeta_1 \times \zeta_2 = \zeta_2 \times \zeta_1$

— PROOF. Let ζ_1, ζ_2, $\zeta_3 \in \mathbb{C}$ be any two complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$, and $\zeta_3 = \xi_3 + i\eta_3$, where ξ_1, η_1, ..., ξ_3, $\eta_3 \in \mathbb{C}$. Then,

$$\begin{cases} \zeta_1 + \zeta_2 = (\xi_1 + \xi_2) + i(\eta_1 + \eta_2) = (\xi_2 + \xi_1) + i(\eta_2 + \eta_1) = \zeta_2 + \zeta_1 \\ \zeta_1 \times \zeta_2 = (\xi_1\xi_2 - \eta_1\eta_2) + i(\xi_1\eta_2 + \xi_2\eta_1) = (\xi_2\xi_1 - \eta_2\eta_1) + i(\xi_2\eta_1 + \xi_1\eta_2) = \zeta_2 \times \zeta_1 \end{cases}$$

Thus, $\zeta_1 + \zeta_2 = \zeta_2 + \zeta_1$ and $\zeta_1 \times \zeta_2 = \zeta_2 \times \zeta_1$. Since

$$\begin{aligned}
(\zeta_1 + \zeta_2) + \zeta_3 &= \big((\xi_1 + \xi_2) + \xi_3\big) + i\big((\eta_1 + \eta_2) + \eta_3\big) \\
&= \big(\xi_1 + (\xi_2 + \xi_3)\big) + i\big(\eta_1 + (\eta_2 + \eta_3)\big) = \zeta_1 + (\zeta_2 + \zeta_3)
\end{aligned}$$

then $(\zeta_1 + \zeta_2) + \zeta_3 = \zeta_1 + (\zeta_2 + \zeta_3)$. Since

$$\begin{aligned}
(\zeta_1 \times \zeta_2) \times \zeta_3 &= \big((\xi_1\xi_2 - \eta_1\eta_2) + i\,(\xi_1\eta_2 + \xi_2\eta_1)\big) \times (\xi_3 + i\eta_3) \\
&= (\xi_1\xi_2\xi_3 - \eta_1\eta_2\xi_3 - \xi_1\eta_2\eta_3 - \eta_3\xi_2\eta_1) \\
&+ i\,(\xi_1\eta_2\xi_3 + \xi_3\xi_2\eta_1 + \xi_1\xi_2\eta_3 - \eta_1\eta_2\eta_3) \\
&= \big(\xi_1\,(\xi_2\xi_3 - \eta_2\eta_3) - \eta_1\,(\xi_2\eta_3 + \xi_3\eta_2)\big) \\
&+ i\big(\xi_1\,(\xi_2\eta_3 + \xi_3\eta_2) + \eta_1\,(\xi_2\xi_3 - \eta_2\eta_3)\big) \\
&= \xi_1\,(\xi_2\xi_3 - \eta_2\eta_3) + (i\eta)\,\big(i\,(\xi_2\eta_3 + \xi_3\eta_2)\big) \\
&+ \xi_1\big(i\,(\xi_2\eta_3 + \xi_3\eta_2)\big) + (i\eta)\,\big((\xi_2\xi_3 - \eta_2\eta_3)\big) \\
&= (\xi_1 + i\eta_1) \times \big((\xi_2\xi_3 - \eta_2\eta_3) + i\,(\xi_2\eta_3 + \xi_3\eta_2)\big) = \zeta_1 \times (\zeta_2 \times \zeta_3)
\end{aligned}$$

Hence, $(\zeta_1 \times \zeta_2) \times \zeta_3 = \zeta_1 \times (\zeta_2 \times \zeta_3)$. Since

$$\begin{aligned}
(\zeta_1 + \zeta_2) \times \zeta_3 &= \big((\xi_1 + \xi_2) + i\,(\eta_1 + \eta_2)\big) \times (\xi_3 + i\eta_3) \\
&= \xi_1\xi_3 + \xi_2\xi_3 + i\eta_1\xi_3 + i\eta_2\xi_3 + i\xi_1\eta_3 + i\xi_2\eta_3 - \eta_1\eta_3 - \eta_2\eta_3 \\
&= \big((\xi_1\xi_3 - \eta_1\eta_3) + i\,(\xi_1\eta_3 + \xi_3\eta_1)\big) + \big((\xi_2\xi_3 - \eta_2\eta_3) + i\,(\xi_2\eta_3 + \xi_3\eta_2)\big) \\
&= (\zeta_1 \times \zeta_3) + (\zeta_2 \times \zeta_3)
\end{aligned}$$

Thus, $(\zeta_1 + \zeta_2) \times \zeta_3 = (\zeta_1 \times \zeta_3) + (\zeta_2 \times \zeta_3)$. Since

$$\begin{aligned}
\zeta_1 \times (\zeta_2 + \zeta_3) &= (\xi_1 + i\eta_1) \times \big((\xi_2 + \xi_3) + i\,(\eta_2 + \eta_3)\big) \\
&= \xi_1\xi_2 + \xi_1\xi_3 + i\xi_1\eta_2 + i\xi_1\eta_3 + i\eta_1\xi_2 + i\eta_1\xi_3 - \eta_1\eta_2 - \eta_1\eta_3 \\
&= \big((\xi_1\xi_2 - \eta_1\eta_2) + i\,(\xi_1\eta_2 + \xi_2\eta_1)\big) + \big((\xi_1\xi_3 - \eta_1\eta_3) + i\,(\xi_1\eta_3 + \xi_1\eta_2)\big) \\
&= (\zeta_1 \times \zeta_2) + (\zeta_1 \times \zeta_3)
\end{aligned}$$

Hence, $\zeta_1 \times (\zeta_2 + \zeta_3) = (\zeta_1 \times \zeta_2) + (\zeta_1 \times \zeta_3)$. The proof of the theorem is, therefore, complete. Q.E.D.

— THEOREM 1.18. *If* ζ_1, $\zeta_2 \in \mathbb{C}$ *be any two complex numbers such that* $\zeta_1 = \xi_1 + i\eta_1$ *and* $\zeta_2 = \xi_2 + i\eta_2$*, then there exist unique solutions* ζ_3, $\zeta_4 \in \mathbb{C}$ *to the following equations:*

– *I.* $\zeta_1 + \zeta_3 = \zeta_2$ – *II.* $\zeta_1 \times \zeta_4 = \zeta_2$

— PROOF. Let ζ_1, $\zeta_2 \in \mathbb{C}$ be complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$, where $\xi_1, \ldots, \eta_2 \in \mathbb{R}$. Suppose ζ_3, $\zeta_4 \in \mathbb{C}$ be some complex numbers such that $\zeta_3 = \xi_3 + i\eta_3$ and $\zeta_4 = \xi_4 + i\eta_4$, satisfying $\zeta_1 + \zeta_3 = \zeta_2$ and $\zeta_1 \times \zeta_4 = \zeta_2$, where $\xi_3, \ldots, \eta_4 \in \mathbb{R}$. Then,

$$\begin{cases} \xi_1 + \xi_3 = \operatorname{Re}(\zeta_1 + \zeta_3) = \operatorname{Re}(\zeta_2) = \xi_2 \\ \eta_1 + \eta_3 = \operatorname{Im}(\zeta_1 + \zeta_3) = \operatorname{Im}(\zeta_2) = \eta_2 \end{cases}$$

Therefore, $\xi_1 + \xi_3 = \xi_2$ and $\eta_1 + \eta_3 = \eta_2$. Hence, $\zeta_3 = (\xi_2 - \xi_1) + i(\eta_2 - \eta_1)$. Since

$$\begin{cases} \xi_1\xi_4 - \eta_1\eta_4 = \operatorname{Re}(\zeta_1 \times \zeta_4) = \operatorname{Re}(\zeta_2) = \xi_2 \\ \xi_1\eta_4 + \xi_4\eta_1 = \operatorname{Im}(\zeta_1 \times \zeta_4) = \operatorname{Im}(\zeta_2) = \eta_2 \end{cases}$$

then $\xi_1\xi_4 - \eta_1\eta_4 = \xi_2$ and $\xi_1\eta_4 + \xi_4\eta_1 = \eta_2$. Consequently, $\xi_4 = \dfrac{\xi_2\xi_1 + \eta_2\eta_1}{\xi_1^2 + \eta_1^2}$ and $\eta_4 = \dfrac{\eta_2\xi_1 - \xi_2\eta_1}{\xi_1^2 + \eta_1^2}$. Thus, $\zeta_4 = \dfrac{\xi_2\xi_1 + \eta_2\eta_1}{\xi_1^2 + \eta_1^2} + i\dfrac{\eta_2\xi_1 - \xi_2\eta_1}{\xi_1^2 + \eta_1^2}$. Therefore, $\zeta_3, \zeta_4 \in \mathbb{C}$ are solutions to $\zeta_1 + \zeta_3 = \zeta_2$ and $\zeta_1 \times \zeta_4 = \zeta_2$, respectively.

Let $\hat{\zeta}_3, \hat{\zeta}_4 \in \mathbb{C}$ such that $\hat{\zeta}_3 = \hat{\xi}_3 + i\hat{\eta}_3$ and $\hat{\zeta}_4 = \hat{\xi}_4 + i\hat{\eta}_4$, where $\hat{\xi}_3, \ldots, \hat{\eta}_4 \in \mathbb{R}$, be other two complex numbers satisfying $\zeta_1 + \hat{\zeta}_3 = \zeta_2$ and $\zeta_1 \times \hat{\zeta}_4 = \zeta_2$. Since

$$\begin{cases} \xi_1 + \hat{\xi}_3 = \operatorname{Re}\big(\zeta_1 + \hat{\zeta}_3\big) = \operatorname{Re}(\zeta_1 + \zeta_3) = \xi_1 + \xi_3 \\ \eta_1 + \hat{\eta}_3 = \operatorname{Im}\big(\zeta_1 + \hat{\zeta}_3\big) = \operatorname{Im}(\zeta_1 + \zeta_3) = \eta_1 + \eta_3 \end{cases}$$

then $\hat{\xi}_3 = \xi_3$ and $\hat{\eta}_3 = \eta_3$. Since

$$\begin{cases} \xi_1\hat{\xi}_4 - \eta_1\hat{\eta}_4 = \operatorname{Re}\big(\zeta_1 \times \hat{\zeta}_4\big) = \operatorname{Re}(\zeta_1 \times \zeta_4) = \xi_1\xi_4 - \eta_1\eta_4 \\ \xi_1\hat{\eta}_4 + \hat{\xi}_4\eta_1 = \operatorname{Im}\big(\zeta_1 \times \hat{\zeta}_4\big) = \operatorname{Im}(\zeta_1 \times \zeta_4) = \xi_1\eta_4 + \xi_4\eta_1 \end{cases}$$

then $\hat{\xi}_4 = \xi_4$ and $\hat{\eta}_4 = \eta_4$. Hence, the solutions $\zeta_3, \zeta_4 \in \mathbb{C}$ are the unique solutions to $\zeta_1 + \zeta_3 = \zeta_2$ and $\zeta_1 \times \zeta_4 = \zeta_2$, respectively. The proof of the theorem is, therefore, complete. Q.E.D.

Evidently, $\zeta_1 - \zeta_2$ and $\zeta_1 \div \zeta_2 = \dfrac{\zeta_1}{\zeta_2}$ are called the *difference between* and *quotient of* the complex numbers $\zeta_1, \zeta_2 \in \mathbb{C}$, respectively.

— DEFINITION 1.19 (*Difference, Quotient*). Let $\mathbb{C} = (\mathcal{C}; +, \times)$ be the complex number system. Then, the binary laws

$$\begin{aligned} -,\ \div :\qquad \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (\xi_1 + i\eta_1,\ \xi_2 + i\eta_2) &\longmapsto (\xi_1 - \xi_2) + i(\eta_1 - \eta_2), \\ &\qquad \frac{\xi_1\xi_2 + \eta_1\eta_2}{\xi_2^2 + \eta_2^2} + i\frac{\eta_1\xi_2 - \xi_1\eta_2}{\xi_2^2 + \eta_2^2} \end{aligned} \tag{1.11}$$

denote the «difference between» and the «quotient of» any two complex numbers $\zeta_1, \zeta_2 \in \mathbb{C}$, respectively.

Suppose ζ_1, $\zeta_2 \in \mathbb{C}$ be any two complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$, where $\xi_1, \ldots, \eta_2 \in \mathbb{R}^2$, then $\zeta_1 = \zeta_2$ obviously implies $\xi_2 = \xi_2$ and $\eta_1 = \eta_2$. Thus, $\zeta_1 - \zeta_2 = 0 + i0$ and $\zeta_1 \div \zeta_2 = 1 + i0$.

— Definition 1.20 (*Null, Unit Elements*). Let $\zeta \in \mathbb{C}$ be any complex number in $\mathbb{C}$. Then, the unique complex numbers $0_{\mathbb{C}}$, $1_{\mathbb{C}} \in \mathbb{C}$, satisfying the following equations

– I. $\zeta + 0_{\mathbb{C}} = \zeta$ – II. $\zeta \times 1_{\mathbb{C}} = \zeta$

is called the «null» and «unit» elements of the complex number system $\mathbb{C}$, respectively.

Suppose ζ_1, $\zeta_2 \in \mathbb{C}$ be any two complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$, where $\xi_1, \ldots, \eta_2 \in \mathbb{R}$, then

$$\begin{aligned} \zeta_1 - \zeta_2 &= \zeta_1 + (-\zeta_2) \\ &= \big(\xi_1 + (-\xi_2)\big) + i\big(\eta_1 + (-\eta_2)\big) \\ &= \big((-\xi_2) + \xi_1\big) + i\big((-\eta_2) + \eta_1\big) = (-\zeta_2) + \zeta_1 \end{aligned}$$

Thus, $\zeta + (-\zeta) = 0_{\mathbb{C}} = (-\zeta) + \zeta$. Clearly, $0_{\mathbb{C}} \longleftrightarrow p(0,0)$ and $1_{\mathbb{C}} \longleftrightarrow p(1,0)$ in the (ξ,η)-plane in $\mathbb{R}^2$. Now, suppose $\zeta \in \mathbb{C}$ be any non-zero complex number such that $\zeta = \xi + i\eta$, where $(\xi,\eta) \in \mathbb{R}^2$, then $\zeta^{-1} = \dfrac{1}{\zeta} = \dfrac{\xi}{\xi^2+\eta^2} - i\dfrac{\eta}{\xi^2+\eta^2}$ and $\zeta \times \zeta^{-1} = 1_{\mathbb{C}}$.

— Definition 1.21 (*Inverse Elements*). Let $\zeta \in \mathbb{C}$ be any complex number in $\mathbb{C}$. Then, the unique complex numbers $-\zeta$, $\zeta^{-1} \in \mathbb{C}$, satisfying the equations

– I. $\zeta + (-\zeta) = 0_{\mathbb{C}} = (-\zeta) + \zeta$ – II. $\zeta \times \zeta^{-1} = 1_{\mathbb{C}} = \zeta^{-1} \times \zeta$

are called the «inverse relative to +» and the «inverse relative to ×» of the complex number ζ.

The following proposition can be proved for complex numbers in $\mathbb{C}$ in the say way as they are usually proved for real numbers in $\mathbb{R}$.

— Proposition 1.22. *If ζ_1, $\zeta_2 \in \mathbb{C}$ be any two complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$, then the following equations hold true:*

– *I.* $\zeta_1 - \zeta_2 + \zeta_2 = \zeta_1$ – *II.* $(\zeta_1 \div \zeta_2) \times \zeta_2 = \zeta_1 \ (\zeta_2 \neq 0)$

– *III.* $\zeta_1 - 0_{\mathbb{C}} = \zeta_1$ – *IV.* $\zeta_1 \div 1_{\mathbb{C}} = \zeta_1$

– *V.* $\zeta_1 - \zeta_1 = 0_{\mathbb{C}}$ – *VI.* $\zeta_1 \div \zeta_1 = 1_{\mathbb{C}} \ (\zeta_1 \neq 0)$

— Proof. Let $\zeta_1, \zeta_2 \in \mathbb{C}$ be any two complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$. Then,

$$\zeta_1 - \zeta_2 + \zeta_2 = \zeta_1 + \big((-\zeta_2) + \zeta_2\big) = \zeta_1 + 0_{\mathbb{C}} = \zeta_1$$

Thus, $\zeta_1 - \zeta_2 + \zeta_2 = \zeta_1$. Since

$$(\zeta_1 \div \zeta_2) \times \zeta_2 = \left(\zeta_1 \times \zeta_2^{-1}\right) \times \zeta_2 = \zeta_1 \times \left(\zeta_2^{-1} \times \zeta_2\right) = \zeta_1 \times 1_{\mathbb{C}} = \zeta_1$$

then $(\zeta_1 \div \zeta_1) \times \zeta_1 = \zeta_1$. Since

$$\zeta_1 - 0_{\mathbb{C}} = \zeta_1 - \left((-\zeta_1) + \zeta_1\right) = \zeta_1 + \left(\zeta_1 + (-\zeta_1)\right) = \zeta_1$$

then $\zeta_1 - 0_{\mathbb{C}} = \zeta_1$. Since $\zeta_1 \div 1_{\mathbb{C}} = \zeta_1 \times 1_{\mathbb{C}}^{-1} = \zeta_1$, then $\zeta_1 \div 1_{\mathbb{C}} = \zeta_1$. Since $\zeta_1 - \zeta_1 = \zeta_1 + (-\zeta_1) = 0_{\mathbb{C}}$, then $\zeta_1 - \zeta_1 = 0_{\mathbb{C}}$. Since $\zeta_1 \div \zeta_1 = \zeta_1 \times \zeta_1^{-1} = 1_{\mathbb{C}}$, then $\zeta_1 \div \zeta_1 = 1_{\mathbb{C}}$. The proof of the proposition is, therefore, complete. Q.E.D.

The following two propositions can also be proved for complex numbers in $\mathbb{C}$ in the say way as the proofs of the above proposition.

— PROPOSITION 1.23. *If* $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$ *be any three complex numbers such that* $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$, *and* $\zeta_3 = \xi_3 + i\eta_3$, *then the following equations hold true:*

- *I.* $\zeta_1 + (\zeta_2 - \zeta_3) = (\zeta_1 + \zeta_2) - \zeta_3$
- *II.* $\zeta_1 - (\zeta_2 + \zeta_3) = \zeta_1 - \zeta_2 - \zeta_3$
- *III.* $\zeta_1 - (\zeta_2 - \zeta_3) = \zeta_1 - \zeta_2 + \zeta_3$
- *IV.* $\zeta_1 \times (\zeta_2 \div \zeta_3) = (\zeta_1 \times \zeta_2) \div \zeta_3 \;\; (\zeta_3 \neq 0)$
- *V.* $\zeta_1 \div (\zeta_2 \times \zeta_3) = (\zeta_1 \div \zeta_2) \div \zeta_3 \;\; (\zeta_2,\ \zeta_3 \neq 0)$
- *VI.* $\zeta_1 \div (\zeta_2 \div \zeta_3) = (\zeta_1 \div \zeta_2) \times \zeta_3 \;\; (\zeta_2,\ \zeta_3 \neq 0)$

— PROOF. Let $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$ be any three complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$, and $\zeta_3 = \xi_3 + i\eta_3$. Then,

$$\zeta_1 + (\zeta_2 - \zeta_3) = \zeta_1 + \left(\zeta_2 + (-\zeta_3)\right) = (\zeta_1 + \zeta_2) + (-\zeta_3) = (\zeta_1 + \zeta_2) - \zeta_3$$

Hence, $\zeta_1 + (\zeta_2 - \zeta_3) = (\zeta_1 + \zeta_2) - \zeta_3$. Since

$$\begin{aligned}\zeta_1 - (\zeta_2 + \zeta_3) &= \zeta_1 + \left(-(\zeta_2 + \zeta_3)\right) \\ &= \zeta_1 + \left((-\zeta_2) + (-\zeta_3)\right) = \zeta_1 + (-\zeta_2) + (-\zeta_3) = \zeta_1 - \zeta_2 - \zeta_3\end{aligned}$$

then $\zeta_1 - (\zeta_2 + \zeta_3) = \zeta_1 - \zeta_2 - \zeta_3$. Since

$$\begin{aligned}\zeta_1 - (\zeta_2 - \zeta_3) &= \zeta_1 + \left(-(\zeta_2 - \zeta_3)\right) \\ &= \zeta_1 + \left((-\zeta_2) + (-(-\zeta_2))\right) = \zeta_1 + (-\zeta_2) + \zeta_3 = \zeta_1 - \zeta_2 - \zeta_3\end{aligned}$$

then $\zeta_1 - (\zeta_2 - \zeta_3) = \zeta_1 - \zeta_2 + \zeta_3$. Since

$$\zeta_1 \times (\zeta_2 \div \zeta_3) = \zeta_1 \times \left(\zeta_2 \times \zeta_3^{-1}\right) = \left(\zeta_1 \times \zeta_2\right) \times \zeta_3^{-1} = \left(\zeta_1 \times \zeta_2\right) \div \zeta_3$$

then $\zeta_1 \times (\zeta_2 \div \zeta_3) = (\zeta_1 \times \zeta_2) \div \zeta_3$. Since

$$\begin{aligned}\zeta_1 \div (\zeta_2 \times \zeta_3) &= \zeta_1 \times (\zeta_2 \times \zeta_3)^{-1} \\ &= \zeta_1 \times \left(\zeta_2^{-1} \times \zeta_3^{-1}\right) = \left(\zeta_1 \times \zeta_2^{-1}\right) \times \zeta_3^{-1} = (\zeta_1 \div \zeta_2) \div \zeta_3\end{aligned}$$

then $\zeta_1 \div (\zeta_2 \times \zeta_3) = (\zeta_1 \div \zeta_2) \div \zeta_3$. Since

$$\begin{aligned}\zeta_1 \div (\zeta_2 \div \zeta_3) &= \zeta_1 \times \left(\zeta_2 \times \zeta_3^{-1}\right)^{-1} \\ &= \zeta_1 \times \left(\zeta_2^{-1} \times \zeta_3\right) = \left(\zeta_1 \times \zeta_2^{-1}\right) \times \zeta_3 = (\zeta_1 \div \zeta) \times \zeta_3\end{aligned}$$

then $\zeta_1 \div (\zeta_2 \div \zeta_3) = (\zeta_1 \div \zeta_2) \times \zeta_3$. The proof of the proposition is, therefore, complete. Q.E.D.

— ILLUSTRATIVE EXAMPLE 1.24. Let ζ_1, ζ_2, $\zeta_3 \in \mathbb{C}$ be three complex numbers such that $\zeta_1 = 3\alpha + i7\beta$, $\zeta_2 = 5\alpha + i9\beta$, and $\zeta_3 = 7\alpha + i11\beta$, where $(\alpha, \beta) \in \mathbb{R}^2$. Show that the equation $\zeta_1 + (\zeta_2 - \zeta_3) = (\zeta_1 + \zeta_2) - \zeta_3$ hold true.

– SOLUTION. The calculation of $\zeta_1 + (\zeta_2 - \zeta_3)$ gives

$$\begin{aligned}\zeta_1 + (\zeta_2 - \zeta_3) &= (3\alpha + i7\beta) + \big((5\alpha + i9\beta) - (7\alpha + i11\beta)\big) \\ &= (3\alpha + i7\beta) + i\,(-2\alpha - i2\beta) = \alpha - i5\beta\end{aligned}$$

and that of $(\zeta_1 + \zeta_2) - \zeta_3$ gives

$$\begin{aligned}(\zeta_1 + \zeta_2) - \zeta_3 &= \big((3\alpha + i7\beta) + (5\alpha + i9\beta)\big) - (7\alpha + i11\beta) \\ &= (8\alpha + i16\beta) - (7\alpha + i11\beta) \\ &= (8\alpha - 7\alpha) + i\,(16\beta - 11\beta) = \alpha + i5\beta\end{aligned}$$

The equation $\zeta_1 \times (\zeta_2 - \zeta_3) = (\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3)$, therefore, holds true.

— PROPOSITION 1.25. *If ζ_1, ζ_2, $\zeta_3 \in \mathbb{C}$ be any three complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$, and $\zeta_3 = \xi_3 + i\eta_3$, then the following equations hold true:*

- *I.* $\zeta_1 \times (\zeta_2 - \zeta_3) = (\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3)$
- *II.* $(\zeta_1 + \zeta_2) \div \zeta_3 = (\zeta_1 \div \zeta_3) + (\zeta_2 \div \zeta_3)$ $(\zeta_3 \neq 0)$
- *III.* $(\zeta_1 - \zeta_2) \div \zeta_3 = (\zeta_1 \div \zeta_3) - (\zeta_2 \div \zeta_3)$ $(\zeta_3 \neq 0)$
- *IV.* $(\zeta_1 \times \zeta_3) \div (\zeta_2 \times \zeta_3) = \zeta_1 \div \zeta_2$ $(\zeta_2,\ \zeta_3 \neq 0)$

— PROOF. Let ζ_1, ζ_2, $\zeta_3 \in \mathbb{C}$ be any three complex numbers such that $\zeta_1 = \xi_1 + i\eta_1$, $\zeta_2 = \xi_2 + i\eta_2$, and $\zeta_3 = \xi_3 + i\eta_3$. Then,

$$\begin{aligned}
\zeta_1 \times (\zeta_2 - \zeta_3) &= \zeta_1 \times \big(\zeta_2 + (-\zeta_3)\big) \\
&= (\zeta_1 \times \zeta_2) + \big(\zeta_1 \times (-\zeta_3)\big) \\
&= (\zeta_1 \times \zeta_2) + \big(-(\zeta_1 \times \zeta_3)\big) = (\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3)
\end{aligned}$$

Thus, $\zeta_1 \times (\zeta_2 - \zeta_3) = (\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3)$. Since

$$\begin{aligned}
(\zeta_1 + \zeta_2) \div \zeta_3 &= (\zeta_1 + \zeta_2) \times \zeta_3^{-1} \\
&= \big(\zeta_1 \times \zeta_3^{-1}\big) + \big(\zeta_2 \times \zeta_3^{-1}\big) = (\zeta_1 \div \zeta_3) + (\zeta_2 \div \zeta_3)
\end{aligned}$$

then $(\zeta_1 + \zeta_2) \div \zeta_3 = (\zeta_1 \div \zeta_3) + (\zeta_2 \div \zeta_3)$. Since

$$\begin{aligned}
(\zeta_1 - \zeta_2) \div \zeta_3 &= (\zeta_1 - \zeta_2) \times \zeta_3^{-1} \\
&= \big(\zeta_1 \times \zeta_3^{-1}\big) - \big(\zeta_2 \times \zeta_3^{-1}\big) = (\zeta_1 \div \zeta_3) - (\zeta_2 \div \zeta_3)
\end{aligned}$$

then $(\zeta_1 - \zeta_2) \div \zeta_3 = (\zeta_1 \div \zeta_3) - (\zeta_2 \div \zeta_3)$. Since

$$\begin{aligned}
(\zeta_1 \times \zeta_3) \div (\zeta_2 \times \zeta_3) &= (\zeta_1 \times \zeta_3) \times (\zeta_2 \times \zeta_3)^{-1} \\
&= (\zeta_1 \times \zeta_3) \times \big(\zeta_2^{-1} \times \zeta_3^{-1}\big) \\
&= (\zeta_1 \times \zeta_3) \times \big(\zeta_3^{-1} \times \zeta_2^{-1}\big) \\
&= \zeta_1 \times \big(\zeta_3 \times \zeta_3^{-1}\big) \times \zeta_2^{-1} \\
&= \zeta_1 \times 1_{\mathbb{C}} \times \zeta_2^{-1} = \zeta_1 \times \zeta_2^{-1} = \zeta_1 \div \zeta_2
\end{aligned}$$

then $(\zeta_1 \times \zeta_3) \div (\zeta_2 \times \zeta_3) = \zeta_1 \div \zeta_2$. The proof of the proposition is, therefore, complete. Q.E.D.

— ILLUSTRATIVE EXAMPLE 1.26. Let ζ_1, ζ_2, $\zeta_3 \in \mathbb{C}$ be three complex numbers such that $\zeta_1 = \alpha + i2\beta$, $\zeta_2 = 3\alpha + i7\beta$, and $\zeta_3 = 4\alpha + i9\beta$, where $(\alpha, \beta) \in \mathbb{R}^{*2}$. Show that the equation $\zeta_1 \times (\zeta_2 - \zeta_3) = (\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3)$ hold true.

– SOLUTION. Since $(\alpha, \beta) \in \mathbb{R}^{*2}$, it is evident that ζ_1, ζ_2, $\zeta_3 \neq 0$. The calculation of $\zeta_1 \times (\zeta_2 - \zeta_3)$ gives

$$\begin{aligned}
\zeta_1 \times (\zeta_2 - \zeta_3) &= (\alpha + i2\beta) \times \big((3\alpha - 4\alpha) + i\,(7\beta - 9\beta)\big) \\
&= (\alpha + i2\beta) \times (-\alpha - i2\beta) = \big(-\alpha^2 + 4\beta^2\big) - i4\alpha\beta
\end{aligned}$$

and that of $(\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3)$ gives

$$\begin{aligned} (\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3) &= \big((\alpha + i2\beta) \times (3\alpha + i7\beta)\big) - \big((\alpha + i2\beta) \times (4\alpha + i9\beta)\big) \\ &= \big(\big(3\alpha^2 - 14\beta^2\big) + 13i\alpha\beta\big) - \big(\big(4\alpha^2 - 18\beta^2\big) + 17i\alpha\beta\big) \\ &= \big(\big(3\alpha^2 - 14\beta^2\big) - \big(4\alpha^2 - 18\beta^2\big)\big) + i\big(+13\alpha\beta - 17i\alpha\beta\big) \\ &= \big(-\alpha^2 + 4\beta^2\big) - i4\alpha\beta \end{aligned}$$

The equation $\zeta_1 \times (\zeta_2 - \zeta_3) = (\zeta_1 \times \zeta_2) - (\zeta_1 \times \zeta_3)$, therefore, holds true.

— Corollary 1.27 (*Field*). *Let* $\mathbb{C} = (\mathcal{C}; +, \times)$ *be the complex number system. Then,* $\mathbb{C}$ *satisfies the following «field axioms»:*

- $\mathrm{Ax}_1\,(\mathbb{C})$ *For every* $(\xi, \eta, \zeta) \in \mathbb{C}^3$, $\xi + (\eta + \zeta) = (\xi + \eta) + \zeta$.

- $\mathrm{Ax}_2\,(\mathbb{C})$ *For every* $(\xi, \eta) \in \mathbb{C}^2$, $\xi + \eta = \eta + \xi$.

- $\mathrm{Ax}_3\,(\mathbb{C})$ *For every* $\xi \in \mathbb{C}$, *there is a unique null element* $0_{\mathbb{C}} \in \mathbb{C}$ *such that* $\xi + 0_{\mathbb{C}} = \xi = 0_{\mathbb{C}} + \xi$.

- $\mathrm{Ax}_4\,(\mathbb{C})$ *For every element* $\xi \in \mathbb{C}$, *there is a unique inverse* $-\xi \in \mathbb{C}$ *relative to the law* $+$ *such that* $\xi + (-\xi) = 0_{\mathbb{C}} = (-\xi) + \xi$.

- $\mathrm{Ax}_5\,(\mathbb{C})$ *For every* $(\xi, \eta, \zeta) \in \mathbb{C}^3$, $\xi \times (\eta \times \zeta) = (\xi \times \eta) \times \zeta$.

- $\mathrm{Ax}_6\,(\mathbb{C})$ *For every* $(\xi, \eta, \zeta) \in \mathbb{C}^3$, $\xi \times (\eta + \zeta) = (\xi \times \eta) + (\xi \times \zeta)$ *and* $(\eta + \zeta) \times \xi = (\eta \times \xi) + (\zeta \times \xi)$.

- $\mathrm{Ax}_7\,(\mathbb{C})$ *For every* $(\xi, \eta) \in \mathbb{C}^2$, $\xi \times \eta = \eta \times \xi$. *For every* $\xi \in \mathbb{C}$, *there is a unique unit element* $0_{\mathbb{C}} \in \mathbb{C}$ *such that* $\xi \times 1_{\mathbb{C}} = \xi = 1_{\mathbb{C}} \times \xi$.

- $\mathrm{Ax}_8\,(\mathbb{C})$ *For every non-zero element* $\xi \in \mathbb{C}$, *there is a unique inverse* $\xi^{-1} = \dfrac{1}{\xi} \in \mathbb{C}$ *relative to the law* $\times$ *such that* $\xi \times \xi^{-1} = 1_{\mathbb{C}} = \xi^{-1} \times \xi$.

— § 1.4. Concluding Remarks

— § 1.5. Selected Problems

The following problems form an integral part of Lesson № 1 on the complex number system $\mathbb{C}$ and arithmetic operations on $\mathbb{C}\times\mathbb{C}$. Many of these problems are routine in nature. Others are more demanding. A few provide examples that are considered challenging, interesting, and instructive.

— Learning Outcomes. After doing Assignment 1.1, the student will be able to:

- – I. understand how quadratic equations in $\left\{\alpha z^2+\beta z+\gamma = 0 :\ (\alpha,\beta,\gamma)\, R^3\right\}$ lead to the field $\mathbb{C} = \left(\left\{z = x + yi :\ (x,y) \in \mathbb{R}^2\right\}; +, \times\right)$ of complex numbers and how to plot complex numbers in $\mathbb{C}$ on a Argand diagram in $\mathbb{R}\times\mathbb{R}$,
- – II. perform basic arithmetic operations on $\mathbb{C}\times\mathbb{C}$ and interpret the results in Cartesian, polar and exponential forms,
- – III. explain Euler's formula $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ and the exponential form $z = r\exp(i\theta)$ of a complex number $z = x + yi$,
- – IV. use de Moivre's formula $\Big(r\big(\cos(\theta) + i\sin(\theta)\big)\Big)^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big)$.

— Problem 1.1 (*Complex Numbers*). Find the values of the following complex numbers:

– I. $(1+2i)^3$ – II. $\dfrac{5}{-3+4i}$ – III. $\left(\dfrac{2+i}{3-2i}\right)^2$ – IV. $(1+i)^n + (1-i)^n$

— Problem 1.2 (*Complex Numbers*). Find the values of the following complex numbers:

– I. $\sqrt{i}$ – II. $\sqrt{-i}$ – III. $\sqrt{1+i}$ – IV. $\sqrt{\dfrac{1-i\sqrt{3}}{2}}$

— Problem 1.3 (*Complex Numbers*). Find the absolute values of the following complex numbers:

– I. $-2i(3+i)(2+4i)(1+i)$ – II. $\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$

— Problem 1.4 (*Complex Numbers*). Let $\zeta \in \mathcal{C}$ be a complex number such that $\zeta = \xi + i\eta = \mathrm{Re}(\zeta) + i\,\mathrm{Im}(\zeta)$. Find the real and imaginary parts in each of the following equations:

– I. $(1-2i)\,\xi + (1+2i)\,\eta = 1 + i$ – II. $\dfrac{\xi-3}{3+i} + \dfrac{\eta-3}{3-i} = i$

– III. $(4-3i)\,\xi^2 + (3+2i)\,\xi\eta = 4\eta^2 - \dfrac{1}{2}\xi^2 = \left(3\xi\eta - 2\eta^2\right) i$

— PROBLEM 1.5 (*Complex Numbers*). Find the geometric interpretation for the following complex numbers:

– I. $\zeta_1 = 3 + i$ – II. $\zeta_2 = -4 + 2i$ – III. $\zeta_3 = -5 - 4i$ – IV. $\zeta_4 = 5 - i$

– V. $\zeta_5 = 1$ – VI. $\zeta_6 = -3i$ – VII. $\zeta_7 = 2i$ – VIII. $\zeta_8 = -4$

— PROBLEM 1.6 (*Complex Numbers*). Find the geometric interpretation for the following equations:

– I. $(-5 + 4i) + (2 - 3i) = -3 + i$ – II. $(4 - i) + (-6 + 4i) = -2 + 3i$

– III. $(-3 - 2i) - (5 + 3i) = 2 - 3i$ – IV. $(8 - i) - (5 + 3i) = 3 - 4i$

– V. $2\,(-4 + 2i) = -8 + 4i$ – VI. $-\,3\,(-1 + 2i) = 3 - 6i$

— PROBLEM 1.7 (*Complex Numbers*). Let ζ_1, ζ_2, $\zeta_3 \in \mathcal{C}$ be any two complex numbers such that $\zeta_1 = 1+2i$, $\zeta_2 = -2+3i$, and $\zeta_3 = 1-i$. Find the following complex numbers:

– I. $\zeta_1 + \zeta_2 + \zeta_3$ – II. $\zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1$ – III. $\zeta_1\zeta_2\zeta_3$

– IV. $\zeta_1^2 + \zeta_2^2 + \zeta_3^2$ – V. $\dfrac{\zeta_1}{\zeta_2} + \dfrac{\zeta_2}{\zeta_3} + \dfrac{\zeta_3}{\zeta_1}$ – VI. $\dfrac{\zeta_1^2 + \zeta_2^2}{\zeta_2^2 + \zeta_3^2}$

— PROBLEM 1.8 (*Complex Numbers*). Evaluate the following complex numbers in the form $\zeta = \xi + i\eta$, where $(\xi, \eta) \in \mathbb{R}^2$:

– I. $(1 + 2i)\,(3 + 4i)$ – II. $\dfrac{1}{3 + 2i}$ – III. $(2 + i)^4$

– IV. $\dfrac{1}{(4 + 2i)\,(2 - 3i)}$ – V. $\dfrac{3 + 4i}{1 + 2i}$ – VI. $\dfrac{1 - i}{1 + i}$

— PROBLEM 1.9 (*Complex Numbers*). Find the modulus for ζ the following equations:

– I. $(\alpha_1 + i\beta_1)^2$ – II. $\dfrac{1}{(\alpha_2 + i\beta_2)^2}$

– III. $\sqrt{\alpha_3 + i\beta_3}$ – IV. $\dfrac{1}{\sqrt{\alpha_4 + i\beta_4}}$

— PROBLEM 1.10 (*Complex Numbers*). Solve the following equations for $\zeta = \xi + i\eta$, where $(\xi, \eta) \in \mathbb{R}^2$:

– I. $(2 + i)\,\zeta + i = 3$ – II. $\dfrac{\zeta - 1}{\zeta - i} = \dfrac{2}{3}$

– III. $\zeta^2 - (3 + i)\,\zeta + 4 + 3i = 0$ – IV. $\zeta^4 - 2\zeta^2 + 4 = 0$