— LESSON № 4 —

THE DIFFERENTIAL OPERATORS $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$ AND THE PRINCIPLES OF DIFFERENTIATION

Le mérite de Georg Cantor est de hasarder dans le domaine de l'infini, sans craindre la lutte intérieure ni extérieure, non seulement avec les paradoxes imaginaires, les préjugés largement répandus, les sentences des philosophes, mais aussi avec le parti pris du nombre de grands mathématiciens. Ainsi il créa une nouvelle science, la théorie des ensembles.

— *FELIX HAUSDORFF (1927)*

— OVERVIEW. Many important properties of a function of a variable are found with the aid of a related function called the *derivative of the function with respect to the variable*. Therefore, knowledge of the derivation of functions is clearly absolutely essential for further studies in so many engineering disciplines. LESSON № 4 introduces the definition and the basic properties of the *differential operators* $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$. This is followed by a presentation of the *principles of differentiation*.

— KEY WORDS AND PHRASES. *Average and exact rates, differential operators, derivative, geometric and physical meaning of the derivative, derivatives of elementary functions, properties of the derivative*

— LEARNING OUTCOMES. After doing ASSIGNMENT 1.1, the student will be able to:

– I. understand the notions of increment, difference, inexact and exact differentials, differential quotient, differential operators $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$ and their fundamental properties,

– II. perform basic differential operations using $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$ to establish average and exact rates, derivatives of elementary functions

– III. explain the behaviors of functions through the differential operators $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$,

– IV. use the differential operators $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$ from first principles.

— § 4.1. Introduction

The object of the Differential Calculus is to compare with a change in the argument of a function the corresponding change in the value of the function or, as we may say, to measure the *rate of change* of a function, that is the average change in value per unit change of argument. For a change in argument from ξ to ζ the value changes from $\phi(\xi)$ to $\phi(\zeta)$, and the ratio of the amount by which the value changes to the amount of change in the argument is $\dfrac{\phi(\zeta)-\phi(\xi)}{\zeta-\xi}$. If for each value of ξ, we can determine a number $\mathrm{D}_\xi\,\phi(\xi)$, say, such that $\dfrac{\phi(\zeta)-\phi(\xi)}{\zeta-\xi}$ differs from $\mathrm{D}_\xi\,\phi(\xi)$ by as little as we please for all values of ζ *sufficiently near* to ξ, the function $\phi(\xi)$ is said to be *regular* or *differentiable*, and $\mathrm{D}_\xi\,\phi(\xi)$ is called the *derivative* of $\phi(\xi)$. Therefore, $\mathrm{D}_\xi\,\phi(\xi)$ records the rate at which $\phi(\xi)$ is changing its value at the point ξ. For instance, since $\dfrac{\zeta^2-\xi^2}{\zeta-\xi}=\zeta+\xi$ and since $\zeta+\xi$ is nearly equal to $\xi+\xi=2\xi$, when ζ is near to ξ, it follows that 2ξ is the derivative of ξ^2. Thus the greater the value of ξ, the faster does ξ^2 change its value. At the point $\xi=5$ the function ξ^2 is increasing 10 times as fast as its argument and at $\xi=20$, ξ^2 is increasing 40 times as fast as its argument. Observe that in considering the ratio $\dfrac{\zeta^2-\xi^2}{\zeta-\xi}$ we are concerned with values of ζ *close* to ξ but *different from* ξ. When we say that the derivative of ξ^2 is 2ξ because $\zeta+\xi$ is nearly equal to 2ξ, the fact that $\zeta+\xi$ actually equals 2ξ when ζ equals ξ is quite irrelevant, because the *only* values of ζ which we are considering are those near ξ and different from it, and $\dfrac{\zeta^2-\xi^2}{\zeta-\xi}=\zeta+\xi$ is equal to $\zeta+\xi$ only so long as ζ and ξ are unequal.

We do not attempt to measure the rate of change of the value of functions as $\phi(\xi)$ for which $\dfrac{\phi(\zeta)-\phi(\xi)}{\zeta-\xi}$ varies appreciably for small variations in ζ.

Another way in which we might express the condition that a function be regular is to say that for small enough changes in the argument, changes in the value of the function are nearly proportional to changes in the argument, that is, if ζ, η are points near ξ, then $\dfrac{\phi(\zeta)-\phi(\xi)}{\phi(\eta)-\phi(\xi)}$ is nearly equal to $\dfrac{\zeta-\xi}{\eta-\xi}$.

The student who is familiar with coordinate geometry will recognize that the condition for a function $\phi(\xi)$ to be regular is just that any sufficiently small part of the curve $\zeta=\phi(\xi)$ is nearly a straight line. Any three points (ξ_1,ζ_1), (ξ_2,ζ_2), (ξ_3,ζ_3) on a straight line satisfy the condition $\dfrac{\zeta_1-\zeta}{\zeta_2-\zeta}=\dfrac{\xi_1-\xi}{\xi_2-\xi}$. For a differentiable function $\phi(\xi)$, if the points lie on $\zeta = \phi(\xi)$, the ratio $\dfrac{\zeta_1 - \zeta}{\zeta_2 - \zeta}$ is *nearly* equal to $\dfrac{\xi_1 - \xi}{\xi_2 - \xi}$, provided ξ_1 and ξ_2 are both close to ξ.

— § 4.2. The Differential Operators $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$

4.2.1. Increments. When a variable changes from ξ to $\xi + \delta\xi$, the quantity $\delta\xi < 0$ is said to express an algebraic decrease while $\delta\xi > 0$ is said to express an algebraic increase in ξ. In either situations, $\delta\xi$ is called an *increment* in the variable ξ. Let $\zeta = \phi(\xi)$, then (ξ, ζ) forms a pair of *independent* and *dependent* variables. When ξ receives an increment $\delta\xi$, the increment $\delta\zeta = \delta\phi(\xi)$ will be determined. The increments $\delta\xi$ and $\delta\zeta = \delta\phi(\xi)$ of ξ, ζ, respectively, thus correspond.

— Definition 4.1 (*Increments*). Let $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$. Then:

– i. $\delta\xi = \xi - (\xi + \delta\xi)$ is called the «increment of the variable ξ»,

– ii. $\delta\phi(\xi) = \phi(\xi + \delta\xi) - \phi(\xi)$ is called the «increment of the function ϕ».

— Illustrative Example 4.2. Let $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function defined on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ by $\phi(\xi) = \alpha\xi^2 + \beta\xi + \gamma$, where $(\alpha, \beta, \gamma) \in \mathbb{R}^3$. Calculate the increment of ϕ.

Solution. First assign the variable ξ an increment $\delta\xi$. Then,

$$\begin{aligned}\delta\phi(\xi) &= \phi(\xi + \delta\xi) - \phi(\xi) \\ &= \left(\alpha(\xi + \delta\xi)^2 + \beta(\xi + \delta\xi) + \gamma\right) - \left(\alpha\xi^2 + \beta\xi + \gamma\right) \\ &= (2\alpha\xi + \beta)\,\delta\xi + \alpha\delta\xi^2\end{aligned}$$

which is a function of ξ, $\delta\xi$, and $\delta\xi^2$.

4.2.2. Average Rate of Change. Let $\zeta = \phi(\xi)$ be given. When an increment $\delta\xi$ is assigned to the independant variable ξ, it produces a corresponding increment $\delta\zeta = \delta\phi(\xi)$ in the dependant variable ζ. The ratio $\dfrac{\delta\phi}{\delta\xi}(\xi)$ describes the *average rate of change of ϕ with respect to ξ* in the given interval $\mathscr{I}_{(\xi,\xi+\delta\xi)}$.

— Definition 4.3 (*Average Rate of Change*). Let $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ be a function on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$. Then, the ratio

$$\frac{\delta\phi}{\delta\xi}(\xi) = \frac{\phi(\xi + \delta\xi) - \phi(\xi)}{\delta\xi} \tag{4.1}$$

is called the «average rate of change of the function ϕ with respect to the variable ξ» in the given interval $\mathscr{I}_{(\xi,\xi+\delta\xi)} \subset \mathscr{I}_{(\alpha,\beta)}$.

— Illustrative Example 4.4. Let $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ be a function defined on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ by $\phi(\xi) = \alpha\xi^2 + \beta\xi + \gamma$, where $(\alpha, \beta, \gamma) \in \mathbb{R}^3$. Calculate the average rate of change of ϕ with respect to ξ.

Solution. Since

$$\begin{aligned}\frac{\delta\phi}{\delta\xi}(\xi) &= \frac{\phi(\xi + \delta\xi) - \phi(\xi)}{\delta\xi} \\ &= \frac{(2\alpha\xi + \beta)\,\delta\xi + \alpha\delta\xi^2}{\delta\xi} = (2\alpha\xi + \beta) + \alpha\delta\xi\end{aligned}$$

then $\dfrac{\delta\phi}{\delta\xi}(\xi) = (2\alpha\xi + \beta) + \alpha\delta\xi$, which is a function of ξ and $\delta\xi$.

4.2.3. Order of Infinitesimals. The increment of the independent variable and of the function, when used in the process of differentiation, are infinitesimals.

— Definition 4.5 (*Infinitesimal*). Any increment $\delta\xi$ of a variable quantity $\xi \in \mathbb{R}$ which approaches zero as a limit,

$$\delta\xi \to 0 \tag{4.2}$$

is called an «infinitely small quantity», or simply an «infinitesimal».

An important idea in the use of infinitesimals is that of their *orders*.

— Definition 4.6 (*Order of Infinitesimals*). Let $\begin{array}{rcl}\eta: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \eta(\xi)\end{array}$ be any function of ξ, $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \eta & \longmapsto \phi(\eta)\end{array}$ be any another function but of η, and suppose both functions be infinitesimals. Then:

– i. if $\displaystyle\lim_{\eta\to 0}\left(\frac{\phi\circ\eta}{\eta}\right)(\xi) = \kappa$, then η, ϕ are said to be of the «same order»,

– II. if $\lim_{\eta\to 0}\left(\frac{\phi\circ\eta}{\eta}\right)(\xi) = 0$, then ϕ is said to be of a «higher order» than η,

– III. if $\lim_{\eta\to 0}\left(\frac{\phi\circ\eta}{\eta^n}\right)(\xi) = \kappa$, then ϕ is said to be of the «n^{th} order» with respect to η,

where $\kappa \in \mathbb{R}$ has a finite value not zero.

Suppose $\eta(\xi)$, $(\phi\circ\eta)(\xi)$ are such that $\lim_{\eta\to 0}\left(\frac{\phi\circ\eta}{\eta}\right)(\xi) = \kappa$, where $\kappa \in \mathbb{R}$ has a finite value not zero, then $(\phi\circ\eta)(\xi) = (\kappa\eta)(\xi) + \delta(\eta)$, where $\lim_{\eta\to 0}\delta(\eta) = 0$. Consequently,

$$\lim_{\eta\to 0}\left(\frac{\delta\circ\eta}{\eta}\right)(\xi) = \lim_{\eta\to 0}\left(\frac{\phi\circ\eta}{\eta}\right)(\xi) - \kappa = 0$$

Thus, $\delta(\xi)$ is an infinitesimal of higher order that $\eta(\xi)$.

4.2.4. Derivative Operators. In the present subsection we shall define the *derivative operator* $\mathrm{D}_\xi(\cdot)$ and the *differential operator* $\mathrm{d}(\cdot)$ as well as their operations on functions. That is, if $\phi(\xi)$ is a function defined on $\mathscr{I}_{[\alpha,\beta]}$, we shall give a precise meaning to the corresponding symbols $\mathrm{D}_\xi\,\phi(\xi)$ and $\mathrm{d}\,\phi(\xi)$.

Let $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$. Let $\xi_0 \in \mathscr{I}_{(\alpha,\beta)}$ be a point of $\mathscr{I}_{(\alpha,\beta)}$. Consider the difference quotient $\frac{\phi(\xi_0+h)-\phi(\xi_0)}{h}$ $(h \in \mathbb{R}^*)$ where $\xi_0 + h \in \mathscr{I}_{(\alpha,\beta)}$. If there exists a real number $\mathrm{D}_\xi\,\phi(\xi_0)$ such that given an $\varepsilon \in \mathbb{R}^*_+$, there exists a $\delta(\varepsilon) \in \mathbb{R}^*_+$ such that $\left|\frac{\phi(\xi_0+h)-\phi(\xi_0)}{h} - \mathrm{D}_\xi\,\phi(\xi_0)\right| < \varepsilon$ whenever $0 < |h| < \delta$, then $\phi(\xi)$ is said to have a «derivative» at $\xi_0 \in \mathscr{I}_{(\alpha,\beta)}$ or is «differentiable» at $\xi_0 \in \mathscr{I}_{(\alpha,\beta)}$, and $\mathrm{D}_\xi\,\phi(\xi_0) = \lim_{h\to 0}\frac{\phi(\xi_0+h)-\phi(\xi_0)}{h}$ is called its derivative.

— Definition 4.7 (*Derivative Operator*). The operator

$$\mathrm{D}_\xi(\cdot)(\xi) = \lim_{h\to 0}\frac{(\cdot)(\xi+h)-(\cdot)(\xi)}{h} \tag{4.3}$$

is called the «derivative operator» with respect to the independent variable ξ.

— Postulate 4.8 (*Derivative Operator*). *The derivative operator* $\mathrm{D}_\xi(\cdot)(\xi) = \lim_{h\to 0}\frac{(\cdot)(\xi+h)-(\cdot)(\xi)}{h}$ *satisfies the following properties:*

– *Additivity:* $\mathrm{D}_\xi(\cdot+\cdot)(\xi) = \mathrm{D}_\xi(\cdot)(\xi) + \mathrm{D}_\xi(\cdot)(\xi)$

– *Homogeneity:* $\mathrm{D}_\xi(\cdot)(\kappa\xi) = \kappa\,\mathrm{D}_\xi(\cdot)(\xi)$

for any $(\xi,\eta) \in \mathscr{I}^2_{(\alpha,\beta)}$.

— Definition 4.9 (*Derivative*). Let $\begin{array}{rrcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function defined on $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$. If $\dfrac{\phi(\xi+h)-\phi(\xi)}{h}$ approaches a limit denoted by $\mathrm{D}_\xi\,\phi(\xi)$ as h approaches zero, then the limiting value $\mathrm{D}_\xi\,\phi(\xi)$ is called the «derivative of $\phi(\xi)$ with respect to ξ»:

$$\mathrm{D}_\xi\,\phi(\xi) = \lim_{h\to 0}\frac{\phi(\xi+h)-\phi(\xi)}{h} \tag{4.4}$$

— Illustrative Example 4.10. Let $\begin{array}{rrcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function defined on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ by $\phi(\xi) = \dfrac{\xi+\gamma}{\xi}$, where $\gamma \in \mathbb{R}$. Find the derivative of $\phi(\xi)$ with respect to ξ.

Solution. Since

$$\begin{aligned}\frac{\phi(\xi+h)-\phi(\xi)}{h} &= \frac{\xi+h+\gamma}{\xi+h}\frac{\xi+\gamma}{\xi}\\ &= \frac{(\xi+h+\gamma)\,\xi-(\xi+\gamma)(\xi+h)}{(\xi+h)\,\xi} = -\frac{\gamma}{(\xi+h)\,\xi}\end{aligned}$$

then $\mathrm{D}_\xi\,\phi(\xi) = -\lim\limits_{h\to 0}\dfrac{\gamma}{(\xi+h)\,\xi} = -\dfrac{\gamma}{\xi^2}$, which is a function of ξ.

— Definition 4.11 (*Right, Left-Hand Derivatives*). Let $\begin{array}{rrcl}\phi: & \mathscr{I}_{[\alpha,\beta]} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function on a closed interval $\mathscr{I}_{[\alpha,\beta]} \subset \mathbb{R}$. Then:

– i. $\mathrm{D}_\xi\,\phi(\alpha) = \lim\limits_{h\to 0}\dfrac{\phi(\alpha+h)-\phi(\alpha)}{h}$ is called the «right-hand derivative» at $\xi = \alpha$,

– ii. $\mathrm{D}_\xi\,\phi(\beta) = \lim\limits_{h\to 0}\dfrac{\phi(\beta)-\phi(\beta-h)}{h}$ is called the «left-hand derivative» at $\xi = \beta$.

— Definition 4.12 (1^{st} *Derived Function*). Let $\begin{array}{rrcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function defined on an interval $\mathscr{I}_{(\alpha,\beta)}$. If ϕ has a differential coefficient for all values of $\xi \in \mathscr{I}_{(\alpha,\beta)}$, then $\mathrm{D}_\xi\,\phi(\xi)$ exists for every value $\xi \in \mathscr{I}_{(\alpha,\beta)}$, and is called the «$1^{\text{st}}$ derived function» of $\phi(\xi)$.

A function may be continuous without being differentiable. That is, the mere fact that ϕ is continuous at ξ does not imply that the derivative $\mathrm{D}_\xi\,\phi(\xi)$ exists, that is, the quotient $\dfrac{\phi(\xi+h)-\phi(\xi)}{h}$ may not approach any limit as $h \to 0$. Nut differentiability *does* imply continuity.

— THEOREM 4.13. *If a function* $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ *is differentiable at a particular point, then it is continuous at that point.*

— PROOF. Let $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function that is differentiable as a particular point $\xi = \xi_0$. If $\xi \neq \xi_0$, let $\phi(\xi) = \dfrac{\phi(\xi)-\phi(\xi_0)}{\xi-\xi_0}(\xi-\xi_0)+\phi(\xi_0)$. Using the rules for limits of products and sums, it follows that $\lim\limits_{\xi\to\xi_0}\phi(\xi) = \mathrm{D}_\xi\,\phi(\xi_0)\times 0 + \phi(\xi_0) = \phi(\xi_0)$. The proof of the theorem is, therefore, complete. Q.E.D.

4.2.5. DIFFERENTIAL OPERATORS. In any function as $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$, if $\lim\limits_{\delta\xi\to 0}\dfrac{\delta\phi}{\delta\xi}(\xi) = \mathrm{D}_\xi\,\phi(\xi) \neq 0$ for a given value of ξ, then $\delta\phi(\xi)$, $\delta\xi$ are infinitesimals of the same order for that value of ξ. From the reasoning in §§ 4.2.3, we can write $\delta\phi(\xi) = \mathrm{D}_\xi\,\phi(\xi)\,\delta\xi + \varepsilon(\delta\xi)$, where $\varepsilon(\delta\xi)$ is an infinitesimal of higher order that $\delta\xi$, since $\lim\limits_{\delta\xi\to 0}\dfrac{\varepsilon(\delta\xi)}{\delta\xi} = 0$.

— DEFINITION 4.14. Let $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a given function. Then, the product of the derivative of the function ϕ by the increment $\delta\xi$ of the independent variable ξ, expressed as

$$\mathrm{d}\,\phi(\xi) = \mathrm{D}_\xi\,\phi(\xi)\,\delta\xi \tag{4.5}$$

is known as the «principal part» of the increment $\delta\phi(\xi) = \mathrm{D}_\xi\,\phi(\xi)\,\delta\xi + \varepsilon(\delta\xi)$, where $\varepsilon(\delta\xi)$ is an infinitesimal of higher order than $\delta\xi$, of the function ϕ and $\mathrm{d}\,\phi(\xi)$ is called the «differential of the function ϕ».

— ILLUSTRATIVE EXAMPLE 4.15. Let $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ be a function defined on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ by $\phi(\xi) = \dfrac{\gamma}{\xi}$, where $\gamma \in \mathbb{R}$. When ξ changes from ξ_0 to $\xi_0 + \delta\xi$ find an approximation for the change in $\phi(\xi)$.

SOLUTION. Since the relation $\delta\phi(\xi_0) = \frac{\gamma}{\xi_0 + \delta\xi} - \frac{\gamma}{\xi_0} = -\frac{\gamma\delta\xi}{(\xi_0 + \delta\xi)\,\xi_0}$ and the relation $\mathrm{D}_\xi\,\phi(\xi_0) = \lim_{\delta\xi \to 0} \frac{\delta\phi}{\delta\xi}(\xi_0) = -\frac{\gamma}{\xi_0^2}$ hold true, the exact increment is $\delta\phi(\xi_0) = -\frac{\gamma\delta\xi}{(\xi_0 + \delta\xi)\,\xi_0}$ and the principal part is $\mathrm{D}_\xi\,\phi(\xi_0)\,\delta\xi = -\frac{\gamma\delta\xi}{\xi_0^2}$. By virtue of $\delta\phi(\xi_0) = \mathrm{D}_\xi\,\phi(\xi_0)\,\delta\xi + \varepsilon(\delta\xi)$, it follows that

$$\begin{aligned}\varepsilon(\delta\xi) &= \delta\phi(\xi_0) - \mathrm{D}_\xi\,\phi(\xi_0)\,\delta\xi \\ &= -\frac{\gamma\delta\xi}{(\xi_0 + \delta\xi)\,\xi_0} + \frac{\gamma\delta\xi}{\xi_0^2} = \frac{\gamma\delta\xi}{(\xi_0 + \delta\xi)\,\xi_0^2}\end{aligned}$$

Thus, the principal part $\mathrm{D}_\xi\,\phi(\xi_0)\,\delta\xi = -\frac{\gamma\delta\xi}{\xi_0^2}$ represents the exact increment $\delta\phi(\xi_0) = -\frac{\gamma\delta\xi}{(\xi_0 + \delta\xi)\,\xi_0}$ and the principal part is $\mathrm{D}_\xi\,\phi(\xi_0)\,\delta\xi = -\frac{\gamma\delta\xi}{\xi_0^2}$ with an error $\varepsilon(\delta\xi) = \frac{\gamma\delta\xi}{(\xi_0 + \delta\xi)\,\xi_0^2}$. Note that $\lim_{\delta\xi \to 0} \varepsilon(\delta\xi) = 0$, as was to be expected.

The differential $\mathrm{d}\,\xi$ of the independent variable ξ is the actual increment $\delta\xi$ of ξ, as shown in the following proposition.

— PROPOSITION 4.16. *If ξ be an independent variable defined on an interval $\mathscr{I}_{(\alpha,\beta)}$, then the differential of ξ is equal to its increment:*

$$\mathrm{d}\,\xi = \delta\xi \tag{4.6}$$

— PROOF. Let ξ be an independent variable defined on an interval $\mathscr{I}_{(\alpha,\beta)}$. Suppose $\begin{aligned}\phi:\ \mathscr{I}_{(\alpha,\beta)} &\longrightarrow \mathbb{R} \\ \xi &\longmapsto \phi(\xi)\end{aligned}$ be a function defined on $\mathscr{I}_{(\alpha,\beta)}$. If $\phi(\xi) = \xi$, then

$$\mathrm{d}\,\xi = \mathrm{d}\,\phi(\xi) = \mathrm{D}_\xi\,\phi(\xi)\,\delta\xi = 1 \times \delta\xi$$

Hence, $\mathrm{d}\,\xi = \delta\xi$. The proof of the proposition is, therefore, complete. Q.E.D.

In virtue of the above proposition, it follows that the increment $\delta\xi$ of the independent variable ξ is to be given the same fixed value, which is otherwise arbitrary and of course variable, for all of the several dependent functions $\phi(\xi)$, $\psi(\xi)$, … of ξ which may be under consideration at the same time.

— DEFINITION 4.17 (*Differential Operator*). The operator

$$\mathrm{d}(\cdot)(\xi) = \mathrm{D}_\xi(\cdot)(\xi)\,\mathrm{d}\,\xi \tag{4.7}$$

is called the «differential operator» with respect to the independent variable ξ.

— POSTULATE 4.18 (*Differential Operator*). *The differential operator* $\mathrm{d}(\cdot)(\xi) = \mathrm{D}_\xi(\cdot)(\xi)\,\mathrm{d}\,\xi$ *satisfies the following properties:*

– *Additivity:* $\mathrm{d}(\cdot)(\xi+\eta) = \mathrm{d}(\cdot)(\xi) + \mathrm{d}(\cdot)(\eta)$

– *Homogeneity:* $\mathrm{d}(\cdot)(\kappa\xi) = \kappa\,\mathrm{d}(\cdot)(\xi)$

for any $(\kappa,\xi,\eta) \in \mathbb{R}\times\mathscr{I}^2_{(\alpha,\beta)}$.

The differential $\mathrm{d}\,\phi(\xi)$ of a function $\phi(\xi)$ is the differential coefficient $\mathrm{D}_\xi\,\phi(\xi)$ of the function multiplied by the differential $\mathrm{d}\,\xi$ of the independent variable ξ. The definition follows.

— Definition 4.19 (*Differential*). Let $\begin{array}{rl}\phi: & \mathscr{I}_{(\alpha,\beta)} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \phi(\xi)\end{array}$ be a function defined on $\mathscr{I}_{(\alpha,\beta)}$. Then, the expression

$$\mathrm{d}\,\phi(\xi) = \mathrm{D}_\xi\,\phi(\xi)\,\mathrm{d}\,\xi \tag{4.8}$$

is called the «differential of $\phi(\xi)$ with respect to ξ».

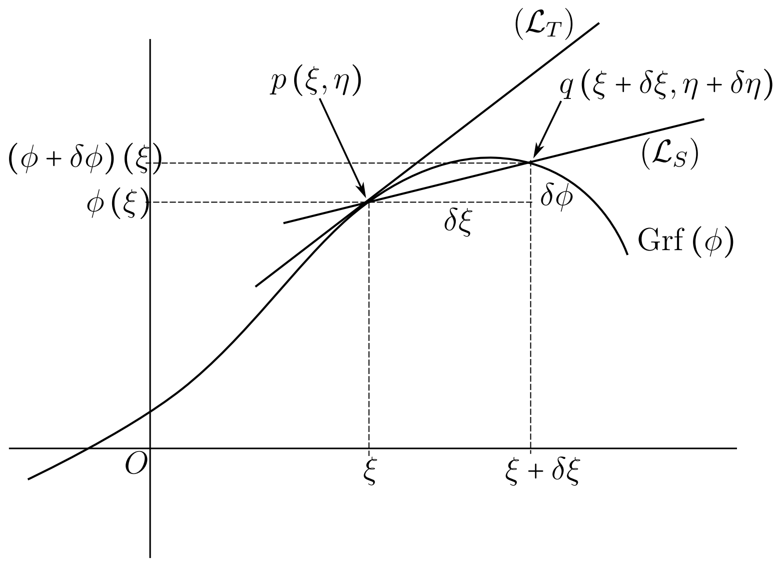
— Postulate 4.20 (*Differential*). *If* $\begin{array}{rl}\phi: & \mathscr{I}_{(\alpha,\beta)} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \phi(\xi)\end{array}$ *be any differentiable function defined on* $\mathscr{I}_{(\alpha,\beta)}\subset\mathbb{R}$*, then the derivative* $\mathrm{D}_\xi\,\phi(\xi)$ *may be regarded as the ratio of the differential* $\mathrm{d}\,\phi(\xi)$ *to the differential* $\mathrm{d}\,\xi$*:*

$$\mathrm{D}_\xi\,\phi(\xi) = \frac{\mathrm{d}\,\phi}{\mathrm{d}\,\xi}(\xi) \tag{4.9}$$

An important *geometrical* and *physical concepts* are associated with the analytic definition of the derivative of a function. These are discussed in the following subsections.

4.2.6. Geometric Derivative. Let $\mathrm{Grf}(\phi) = \left\{\left(\xi,\phi(\xi)\right):\ \xi\in\mathscr{I}_{(\alpha,\beta)}\right\}$ be the graph of a continuous function $\begin{array}{rl}\phi: & \mathscr{I}_{(\alpha,\beta)} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \phi(\xi)\end{array}$ with $p(\xi,\eta)\in\mathrm{Grf}(\phi)$ any point on it, as in Fig. 1. Give ξ an increment $\delta\xi$, then $\dfrac{\delta\phi}{\delta\xi}(\xi) = \dfrac{\phi(\xi+\delta\xi)-\phi(\xi)}{\delta\xi}$ is the slope of the secant line $\mathcal{L}_S$ joining $p(\xi,\eta)$ and $q(\xi+\delta\xi,\eta+\delta\eta)\in\mathrm{Grf}(\phi)$. Let $\delta\xi$ approach zero, then, since $\phi(\xi)$ is a continuous function, $\delta\eta = \delta\phi(\xi)$ approaches zero. That is, the point $q(\xi+\delta\xi,\eta+\delta\eta)$ moves along the graph $\mathrm{Grf}(\phi)$ and approaches $p(\xi,\eta)$ as a limit. Hence if $\dfrac{\delta\phi}{\delta\xi}(\xi)$ has a limit, it is the slope of the limiting position of the secant line to $\mathrm{Grf}(\phi)$. But, the following postulate holds true.

— Postulate 4.21. *Let* $\mathrm{Grf}(\phi) = \left\{\left(\xi,\phi(\xi)\right):\ \xi\in\mathscr{I}_{(\alpha,\beta)}\right\}$ *be the graph of the curve of any function* $\begin{array}{rl}\phi: & \mathscr{I}_{(\alpha,\beta)} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \phi(\xi)\end{array}$. *Then, the «tangent to* $\mathrm{Grf}(\phi)$*» at* *a point* $p(\xi_1, \eta_1) \in \mathrm{Grf}(\phi)$ *is the limiting position of the secant joining* $p(\xi_1, \eta_1)$ *and another point* $q(\xi_2, \eta_2) \in \mathrm{Grf}(\phi)$ *on* $\mathrm{Grf}(\phi)$ *as* $q(\xi_2, \eta_2)$ *approaches* $p(\xi_1, \eta_1)$.



— Figure 1. Illustration of the *Geometric Derivative*.

Therefore, $\lim\limits_{\delta\xi \to 0} \dfrac{\delta\phi}{\delta\xi}(\xi)$ is the slope of the tangent line $\mathcal{L}_T$ to the graph $\mathrm{Grf}(\phi)$ of the function $\phi(\xi)$ at the point $p(\xi_1, \eta_1)$. But, the following postulate holds true. Thus, we can postulate the following statement.

— Postulate 4.22. *Let* $\mathrm{Grf}(\phi) = \{(\xi, \phi(\xi)) : \ \xi \in \mathscr{I}_{(\alpha,\beta)}\}$ *be the graph of the curve of any function* $\begin{array}{rccl} \phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \phi(\xi) \end{array}$. *Then, the numerical value of the derivative* $D_\xi\,\phi(\phi)$ *for any given value* $\xi_0 \in \mathscr{I}_{(\alpha,\beta)}$ *assigned to the variable is the slope of the tangent line* $\mathcal{L}_T$ *at the point on the graph* $\mathrm{Grf}(\phi)$ *of* $\phi(\xi)$ *whose abscissa is* ξ_0.

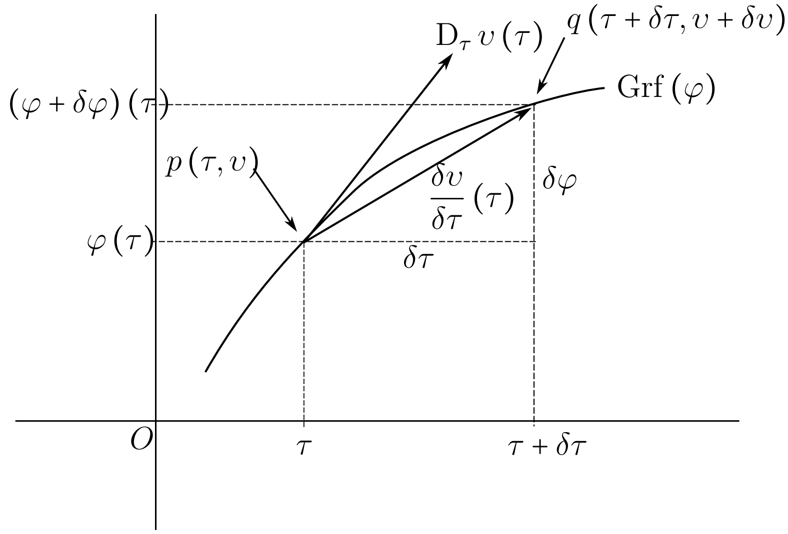
4.2.7. Physical Derivative. Let two physical quantities be connected by a functional relation. Calling them υ, τ and letting υ be a continuous function of τ defined by some functional φ we have, $\upsilon = \varphi(\tau)$. Then any change $\delta\tau$ in the variable τ produces a corresponding change $\delta\upsilon = \delta\varphi(\tau)$ in the variable υ. The postulate follows.

— Postulate 4.23. *Let* $\begin{array}{rccl} \varphi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R} \\ & \tau & \longmapsto & \varphi(\tau) = \upsilon \end{array}$ *be a continuous functional connecting two physical quantities* υ, τ. *Then, the ratio* $\dfrac{\delta\upsilon}{\delta\tau}(\tau) = \dfrac{\delta\varphi}{\delta\tau}(\tau)$ *of* *the increments $\delta\upsilon(\tau)$, $\delta\tau$ is the «average rate of change of $\upsilon = \varphi(\tau)$ with respect to τ in the interval $\mathscr{I}_{(\tau,\tau+\delta\tau)} \subset \mathscr{I}_{(\alpha,\beta)}$».*

Now if we let $\delta\tau$ approach zero, then $\delta\upsilon(\tau)$ approaches the limit zero, since $\upsilon = \varphi(\tau)$ is assumed to be a continuous function of τ on $\mathscr{I}_{(\alpha,\beta)}$. Hence, if the ratio $\dfrac{\delta\upsilon}{\delta\tau}(\tau) = \dfrac{\delta\varphi}{\delta\tau}(\tau)$ has a limit, it is defined to be the rate of change of $\upsilon = \varphi(\tau)$ with respect to τ at the beginning of the interval $\mathscr{I}_{(\tau,\tau+\delta\tau)} \subset \mathscr{I}_{(\alpha,\beta)}$. The postulate follows.

— Postulate 4.24. *Let* $\begin{array}{rccl}\varphi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R} \\ & \tau & \longmapsto & \varphi(\tau) = \upsilon\end{array}$ *be a continuous functional connecting two physical quantities υ, τ. Then, the limit $\mathrm{D}_\tau\,\upsilon(\tau) = \lim\limits_{\delta\tau\to0}\dfrac{\delta\upsilon}{\delta\tau}(\tau)$ is the «exact rate of change of the function $\upsilon = \varphi(\tau)$ with respect to τ in the interval $\mathscr{I}_{(\tau,\tau+\delta\tau)} \subset \mathscr{I}_{(\alpha,\beta)}$» and is measured in units of υ per unit τ.*

An illustration of the such *physical derivative* is presented in Fig. 2, in which $\dfrac{\delta\upsilon}{\delta\tau}(\tau)$ represents the *average rate of change* of $\upsilon = \varphi(\tau)$ with respect to τ while $\mathrm{D}_\tau\,\upsilon(\tau) = \lim\limits_{\delta\tau\to0}\dfrac{\delta\upsilon}{\delta\tau}(\tau)$ represents the *exact rate of change* of the function $\upsilon = \varphi(\tau)$ with respect to τ.



— Figure 2. Illustration of the *Physical Derivative*, *average* and *exact rate of change.*

Thus, the distance $\xi(\tau)$, in meters [m], of a moving object is a function of the time τ, in seconds [s]. An increment of time $\delta\tau$ changes the distance by $\delta\xi(\tau)$.

Then, $\upsilon_{\mathrm{ave}}(\tau) = \dfrac{\delta\xi}{\delta\tau}(\tau)$ is the average rate of change of ξ with respect to τ in some interval $\mathscr{I}_{(\tau,\tau+\delta\tau)} \subset \mathscr{I}_{(\alpha,\beta)}$. This quantity $\upsilon_{\mathrm{ave}}(\tau)$ we call the *average velocity* during the interval $\mathscr{I}_{(\tau,\tau+\delta\tau)}$ and is measured in meters per second, $\left[\mathrm{m}\cdot\mathrm{s}^{-1}\right]$. The value of $\upsilon(\tau) = \mathrm{D}_\tau\,\xi(\tau) = \lim\limits_{\delta\tau\to 0}\dfrac{\delta\xi}{\delta\tau}(\tau)$ for any given $\tau \in \mathscr{I}_{(\alpha,\beta)}$ is the *exact velocity* at that instant in meters per second, $\left[\mathrm{m}\cdot\mathrm{s}^{-1}\right]$.

4.2.8. Antiderivatives. In certain kinds of problems it is required to find all functions whose derivatives is some specific function.

— Definition 4.25 (*Antiderivative*). If $\begin{array}{rrcl}\Phi,\ \phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \Phi(\xi),\ \phi(\xi)\end{array}$ are defined on the same interval $\mathscr{I}_{(\alpha,\beta)}$ of the ξ-axis, and that $\mathrm{D}_\xi\,\Phi(\xi) = \phi(\xi)$. Then $\Phi(\xi)$ is called an «antiderivative» of $\phi(\xi)$.

Finding an antiderivative of a given function is called *antidifferentiation*, because it is the inverse of differentiation. Some functions do not have antiderivatives, for there are functions which cannot be obtained by differentiating other functions. Antiderivatives are not unique, that is, if a functions has one antiderivative, it actually has many antiderivatives, in fact, infinitely many. For if $\Phi(\xi)$ is an antiderivative of $\phi(\xi)$, so is $\Phi(\xi)+\kappa$, where $\kappa \in \mathbb{R}$ is any constant. This is because the derivative of the constant is zero, and $\mathrm{D}_\xi\big(\phi(\xi)+\kappa\big) = \mathrm{D}_\xi\,\Phi(\xi) + \mathrm{D}_\xi\,\kappa = \mathrm{D}_\xi\,\Phi(\xi)$. It is an important fact that all the antiderivatives of a given function can be obtained from a single one of the antiderivatives by adding constants to it.

— Theorem 4.26. *If* $\begin{array}{rrcl}\Phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \Phi(\xi)\end{array}$ *is any particular antiderivative of* $\begin{array}{rrcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi),\end{array}$ *then the expression* $\Phi(\xi)+\kappa$*, where* $\kappa \in \mathbb{R}$ *is an arbitrary constant, represents all possible antiderivatives of* $\phi(\xi)$.

— Proof. Let $\begin{array}{rrcl}\Phi,\ \Psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \Phi(\xi),\ \Psi(\xi)\end{array}$ be any particular antiderivatives of $\begin{array}{rrcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ on $\mathscr{I}_{(\alpha,\beta)}$. Set $\Xi(\xi) = (\Phi-\Psi)(\xi)$. Then,

$$\frac{\Xi(\xi+h)-\Xi(\xi)}{h} = \frac{\Phi(\xi+h)-\Phi(\xi)}{h} - \frac{\Psi(\xi+h)-\Psi(\xi)}{h}$$

So, taking the limits as $h \to 0$, it results that $\mathrm{D}_\xi\,\Xi(\xi) = \mathrm{D}_\xi\,(\Phi-\Psi)(\xi) = 0$. Hence, $\Xi(\xi) = \kappa$, where $\kappa \in \mathbb{R}$ is some constant. But $\Phi(\xi) = (\Psi+\Xi)(\xi)$, so $\Phi(\xi) = \Psi(\xi)+\kappa$, as asserted. The proof of the theorem is, therefore, complete. Q.E.D.

— POSTULATE 4.27 (*General Antiderivative*). *If* $\begin{array}{rl} \Phi: \ \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \Phi(\xi) \end{array}$ *be a particular antiderivative of* $\begin{array}{rl} \phi: \ \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi), \end{array}$ *the expression* $\Phi(\xi)+\kappa$*, where* $\kappa \in \mathbb{R}$ *is an arbitrary constant, is called the «general antiderivative» of* $\phi(\xi)$.

— ILLUSTRATIVE EXAMPLE 4.28. Let $\begin{array}{rl} \phi: \ \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi) \end{array}$ be a function defined on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ by $\phi(\xi) = \alpha\xi^2 + \beta\xi + \gamma$, where $(\alpha,\beta,\gamma) \in \mathbb{R}^3$. Determine the antiderivative of ϕ with respect to ξ.

SOLUTION. Since $\mathrm{D}_\xi\left(\dfrac{\alpha}{3}\xi^3 + \dfrac{\beta}{2}\xi^2 + \gamma\xi + \kappa\right) = \phi(\xi)$, where $\kappa \in \mathbb{R}$ is some constant, $\Phi(\xi) = \dfrac{\alpha}{3}\xi^3 + \dfrac{\beta}{2}\xi^2 + \gamma\xi + \kappa$ is, therefore, the general antiderivative of ϕ with respect to ξ.

— § 4.3. THE PRINCIPLES OF DIFFERENTIATION

The method of establishing the derivative and differential of a function by means of the differential operators $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$, respectively, is perfectly general and can be applied to all differentiable functions. However, by establishing the derivative and differential of a special type of function, by the differential operators $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$, respectively, we obtain a formula which, when memorized, may be used to write down the derivative of any function belonging to that type. Thus, a thorough knowledge of the formulas derived in the following section is essential.

— THEOREM 4.29. *If* $\begin{array}{rl} \phi: \ \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi) = \kappa \end{array}$ *be a constant function, where* $\kappa \in \mathbb{R}$ *is any arbitrary constant, then the derivative of* ϕ *with respect to* ξ *is zero:*

$$\mathrm{D}_\xi\,\phi(\xi) = 0 \tag{4.10}$$

— PROOF. Let $\begin{array}{rl} \phi: \ \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi) = \kappa \end{array}$ be a constant function, where $\kappa \in \mathbb{R}$ is any arbitrary constant. Then, $(\phi + \delta\phi)(\xi) = \kappa$, and $\delta\phi(\xi) = 0$, whence $\dfrac{\phi(\xi+h) - \phi(\xi)}{h} = 0$ and $\mathrm{D}_\xi\,\phi(\xi) = \lim\limits_{h\to 0}\dfrac{\phi(\xi+h)-\phi(\xi)}{h} = 0$. Thus, $\mathrm{D}_\xi\,\phi(\xi) = 0$. The proof of the theorem is, therefore, complete. Q.E.D.

The above theorem is evident if we consider the graph $\mathrm{Grf}(\phi) = \left\{\left(x, \phi(x)\right) : \left(\forall \xi \in \mathscr{I}_{(\alpha,\beta)}\right)\left[\phi(x) = \kappa\right]\right\}$, where $\kappa \in \mathbb{R}$ is some constant, which is a straight line parallel to the ξ-axis. For any points (x,κ), $(x+\delta\xi,\kappa) \in \mathrm{Grf}(\phi)$, the rate of change of ϕ with respect to ξ is zero. In other words, the slope of $\mathrm{Grf}(\phi)$ is always zero.

— Corollary 4.30. *If* $\begin{array}{rrl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi) = \kappa\end{array}$ *be a constant function, where $\kappa \in \mathbb{R}$ is any arbitrary constant, then the differential of ϕ with respect to ξ is zero:*

$$\mathrm{d}\,\phi(\xi) = 0 \tag{4.11}$$

— Theorem 4.31. *If* $\begin{array}{rrl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi) = \xi\end{array}$ *be an identity function, then the derivative of ϕ with respect to ξ is 1:*

$$\mathrm{D}_\xi\,\phi(\xi) = 1 \tag{4.12}$$

— Proof. Let $\begin{array}{rrl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi) = \xi\end{array}$ be an identity function. Then, $(\phi + \delta\phi)(\xi) = \xi + \delta\xi$, and $\delta\phi(\xi) = \delta\xi$, whence $\dfrac{\phi(\xi + h) - \phi(\xi)}{h} = 1$ and $\mathrm{D}_\xi\,\phi(\xi) = \lim\limits_{h\to 0} \dfrac{\phi(\xi + h) - \phi(\xi)}{h} = 1$. Thus, $\mathrm{D}_\xi\,\phi(\xi) = 1$. The proof of the theorem is, therefore, complete. Q.E.D.

— Corollary 4.32. *If* $\begin{array}{rrl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi) = \xi\end{array}$ *be an identity function, then the differential of ϕ with respect to* $\mathrm{d}\,\xi$*:*

$$\mathrm{d}\,\phi(\xi) = \mathrm{d}\,\xi \tag{4.13}$$

— Theorem 4.33. *If* $\begin{array}{rrl}(\kappa\phi): & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto (\kappa\phi)(\xi) = \xi\end{array}$ *be any differentiable function, where $\kappa \in \mathbb{R}$ is any arbitrary constant, then the derivative of the constant κ times the function ϕ with respect to ξ is equal to the constant κ times the derivative of the function ϕ:*

$$\mathrm{D}_\xi\,(\kappa\phi)(\xi) = \kappa\,\mathrm{D}_\xi\,\phi(\xi) \tag{4.14}$$

— Proof. Let $\begin{array}{rrl}(\kappa\phi): & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto (\kappa\phi)(\xi) = \xi\end{array}$ be any function that is differentiable function, where $\kappa \in \mathbb{R}$ is any arbitrary constant. Then, an increment $\delta\xi$ will change $(\kappa\phi)$ to $\big((\kappa\phi) + \delta(\kappa\phi)\big)(\xi) = \kappa(\phi + \delta\phi)(\xi)$. Then, $\delta(\kappa\phi)(\xi) = \kappa\delta\phi(\xi)$, whence $\dfrac{(\kappa\phi)(\xi + h) - (\kappa\phi)(\xi)}{h} = \kappa \times \dfrac{\phi(\xi + h) - \phi(\xi)}{h}$ and,

$$\mathrm{D}_\xi\,(\kappa\phi)(\xi) = \lim_{h\to 0} \kappa \times \frac{\phi(\xi + h) - \phi(\xi)}{h} = \kappa \times \left(\lim_{h\to 0} \frac{\phi(\xi + h) - \phi(\xi)}{h}\right) = \kappa\,\mathrm{D}_\xi\,\phi(\xi)$$

Hence, $\mathrm{D}_\xi\,(\kappa\phi)(\xi) = \kappa\,\mathrm{D}_\xi\,\phi(\xi)$. The proof of the theorem is, therefore, complete. Q.E.D.

— COROLLARY 4.34. *If* $\begin{array}{rcl}(\kappa\phi): & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto (\kappa\phi)(\xi) = \xi\end{array}$ *be any differentiable function, where* $\kappa \in \mathbb{R}$ *is any arbitrary constant, then the differential of the constant* κ *times the function* ϕ *with respect to* ξ *is equal to the constant* κ *times the differential of the function* ϕ*:*

$$\mathrm{d}\,(\kappa\phi)\,(\xi) = \kappa\,\mathrm{d}\,\phi\,(\xi) \tag{4.15}$$

— THEOREM 4.35. *If* $\begin{array}{rcl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ *be any two differentiable functions, then the derivative of the algebraic sum or difference of the functions* ϕ*,* ψ *is the corresponding sum or difference of the derivatives of the functions* ϕ*,* ψ*:*

$$\mathrm{D}_\xi\,(\phi \pm \psi)\,(\xi) = \mathrm{D}_\xi\,\phi\,(\xi) \pm \mathrm{D}_\xi\,\psi\,(\xi) \tag{4.16}$$

— PROOF. Suppose $\begin{array}{rcl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ be any two differentiable functions. Then, an increment $\delta\xi$ will change $(\phi \pm \psi)(\xi)$ to $\big((\phi \pm \psi) + \delta(\phi \pm \psi)\big)(\xi)$. Consequently, $\delta(\phi \pm \psi)(\xi) = \delta\phi(\xi) \pm \delta\psi(\xi)$, whence

$$\frac{(\phi \pm \psi)(\xi + h) - (\phi \pm \psi)(\xi)}{h} = \frac{\phi(\xi + h) - \phi(\xi)}{h} \pm \frac{\psi(\xi + h) - \psi(\xi)}{h}$$

and,

$$\begin{aligned}\mathrm{D}_\xi\,(\phi \pm \psi)\,(\xi) &= \lim_{h\to 0}\left(\frac{\phi(\xi + h) - \phi(\xi)}{h} \pm \frac{\psi(\xi + h) - \psi(\xi)}{h}\right) \\ &= \lim_{h\to 0}\frac{\phi(\xi + h) - \phi(\xi)}{h} \pm \lim_{h\to 0}\frac{\psi(\xi + h) - \psi(\xi)}{h} \\ &= \mathrm{D}_\xi\,\phi\,(\xi) \pm \mathrm{D}_\xi\,\psi\,(\xi)\end{aligned}$$

Hence, $\mathrm{D}_\xi\,(\phi \pm \psi)\,(\xi) = \mathrm{D}_\xi\,\phi\,(\xi) \pm \mathrm{D}_\xi\,\psi\,(\xi)$. The proof of the theorem is, therefore, complete. Q.E.D.

— COROLLARY 4.36. *If* $\begin{array}{rcl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ *be any two differentiable functions, then the differential of the algebraic sum or difference of the functions* ϕ*,* ψ *is the corresponding sum or difference of the differentials of the functions* ϕ*,* ψ*:*

$$\mathrm{d}\,(\phi \pm \psi)\,(\xi) = \mathrm{d}\,\phi\,(\xi) \pm \mathrm{d}\,\psi\,(\xi) \tag{4.17}$$

— PROPOSITION 4.37. *Let* $\begin{array}{rcl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ *be any two differentiable functions on* $\mathscr{I}_{(\alpha,\beta)}$*. If* $\mathrm{D}_\xi\,\phi\,(\xi) = \mathrm{D}_\xi\,\psi\,(\xi)$*, then* ϕ*,* ψ *can only differ by an additive constant* $\kappa \in \mathbb{R}$*.*

— PROOF. Let $\begin{array}{rcl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ be any two differentiable functions on $\in \mathscr{I}_{(\alpha,\beta)}$ such that $\mathrm{D}_\xi\,\phi(\xi) = \mathrm{D}_\xi\,\psi(\xi)$. Set $\Phi(\xi) = (\psi - \phi)(\xi)$. Then, by hypothesis, $\mathrm{D}_\xi\,\Phi(\xi) = \mathrm{D}_\xi\,(\psi - \phi)(\xi) = \mathrm{D}_\xi\,\psi(\xi) - \mathrm{D}_\xi\,\phi(\xi) = 0$. That is, $\mathrm{D}_\xi\,\Phi(\xi) = 0$ for all $\xi \in \mathscr{I}_{(\alpha,\beta)}$. Hence, $\Phi(\xi) = \kappa$, where $\kappa \in \mathbb{R}$. The proof of the proposition is, therefore, complete. Q.E.D.

— THEOREM 4.38. *If* $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ *be any differentiable function and* $n \in \mathbb{Z}_+^*$ *be any positive integer, then:*

$$\mathrm{D}_\xi\big(\phi^n\big)(\xi) = n\phi^{n-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi) \tag{4.18}$$

— PROOF. Let $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ be any differentiable function and $n \in \mathbb{Z}_+^*$ be any positive integer. Then, an increment $\delta\xi$ will change ϕ^n to $\big(\phi^n + \delta\phi^n\big)(\xi)$. Expanding by the Binomial theorem, it follows that

$$\begin{aligned}\big(\phi^n + \delta\phi^n\big)(\xi) &= \phi^n(\xi) + \frac{n}{1!}\phi^{n-1}(\xi)\,\delta\phi(\xi) + \frac{n(n-1)}{2!}\phi^{n-2}(\xi)\big(\delta\phi(\xi)\big)^2 \\ &+ \cdots + \big(\delta\phi(\xi)\big)^n\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\delta\phi^n}{\delta\xi}(\xi) &= \frac{n}{1!}\phi^{n-1}(\xi)\,\frac{\delta\phi}{\delta\xi}(\xi) + \frac{n(n-1)}{2!}\phi^{n-2}(\xi)\,\frac{\delta\phi}{\delta\xi}(\xi)\,\delta\xi \\ &+ \cdots + \frac{\delta\phi}{\delta\xi}(\xi)\big(\delta\phi(\xi)\big)^{n-1}\end{aligned}$$

Taking the limit of both sides as $\delta\xi \to 0$ and keeping in mind that $\lim\limits_{\delta\xi\to 0}\delta\phi(\xi) = 0$, since ϕ is a continuous function of ξ, and that $\mathrm{D}_\xi\,\phi(\xi)$ exists, then $\mathrm{D}_\xi\big(\phi^n\big)(\xi) = \lim\limits_{\delta\xi\to 0}\dfrac{\delta\phi^n}{\delta\xi}(\xi) = n\phi^{n-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)$. Thus, $\mathrm{D}_\xi\big(\phi^n\big)(\xi) = n\phi^{n-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)$. The proof of the theorem is, therefore, complete. Q.E.D.

— COROLLARY 4.39. *If* $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ *be any differentiable function and* $n \in \mathbb{Z}_+^*$ *be any positive integer, then:*

$$\mathrm{d}\big(\phi^n\big)(\xi) = n\phi^{n-1}(\xi)\,\mathrm{d}\,\phi(\xi) \tag{4.19}$$

— PROPOSITION 4.40. *If* $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ *be any differentiable function and* $n \in \mathbb{Z}_-^*$ *be any negative integer, then:*

$$\mathrm{D}_\xi\big(\phi^n\big)(\xi) = n\phi^{n-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi) \tag{4.20}$$

— Proof. Let $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ be any differentiable function and $n \in \mathbb{Z}_-^*$ be any negative integer. Set $n = -m \in \mathbb{Z}_+^*$. Then, $\phi^n(\xi) = \phi^{-m}(\xi)$, implying $(\phi^m \times \phi^n)(\xi) = 1$. Since $(\phi^m \times \phi^n)(\xi)$ is a constant, its derivative with respect to ξ must be zero,

$$\begin{aligned} \mathrm{D}_\xi (\phi^m \times \phi^n)(\xi) &= \phi^m(\xi)\,\mathrm{D}_\xi(\phi^n)(\xi) + \phi^n(\xi)\,\mathrm{D}_\xi(\phi^m)(\xi) \\ &= \phi^m(\xi)\,\mathrm{D}_\xi(\phi^n)(\xi) + m\phi^{m-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)\,\phi^n(\xi) = 0 \end{aligned}$$

Then

$$\begin{aligned} \mathrm{D}_\xi(\phi^n)(\xi) &= -m\frac{\phi^{m-1}(\xi) \times \phi^n(\xi)}{\phi^m(\xi)}\,\mathrm{D}_\xi\,\phi(\xi) \\ &= -m\frac{\phi^{m-1}(\xi) \times \phi^{-m}(\xi)}{\phi^m(\xi)}\,\mathrm{D}_\xi\,\phi(\xi) \\ &= -m\phi^{-m-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi) = n\phi^{n-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi) \end{aligned}$$

Hence, $\mathrm{D}_\xi(\phi^n)(\xi) = n\phi^{n-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)$ for any negative integer $n \in \mathbb{Z}_-^*$. The proof of the proposition is, therefore, complete. Q.E.D.

— Corollary 4.41. *If* $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ *be any differentiable function and* $n \in \mathbb{Z}_-^*$ *be any negative integer, then:*

$$\mathrm{d}\big(\phi^n\big)(\xi) = n\phi^{n-1}(\xi)\,\mathrm{d}\,\phi(\xi) \tag{4.21}$$

— Proposition 4.42. *If* $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ *be any differentiable function and* $q \in \mathbb{Q}$ *be any fractional, then:*

$$\mathrm{D}_\xi\big(\phi^q\big)(\xi) = q\phi^{q-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi) \tag{4.22}$$

— Proof. Let $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ be any differentiable function and $q \in \mathbb{Q}$ be any fractional. Set $q = \dfrac{m}{n}$, where $(m,n) \in \mathbb{Z} \times \mathbb{Z}^*$. Then, $\phi^q(\xi) = \phi^{m/n}(\xi)$, implying $\big(\phi^q\big)^n(\xi) = \phi^m(\xi)$. Since $\big(\phi^q\big)^n(\xi)$ and $\phi^m(\xi)$ are equal functions of ξ, their derivatives with respect to ξ must be equal. Therefore, $\mathrm{D}_\xi\big(\phi^q\big)^n(\xi) =$

$\mathrm{D}_\xi\,\phi^m(\xi)$ or $n\big(\phi^q\big)^{n-1}(\xi)\,\mathrm{D}_\xi\big(\phi^q\big)(\xi) = m\phi^{m-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)$. Consequently,

$$\begin{aligned}\mathrm{D}_\xi\big(\phi^q\big)(\xi) &= \frac{m\phi^{m-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)}{n\big(\phi^q\big)^{n-1}(\xi)} \\ &= \frac{m}{n}\frac{\phi^{m-1}(\xi)}{\big(\phi^{m/n}\big)^{n-1}(\xi)}\,\mathrm{D}_\xi\,\phi(\xi) \\ &= \frac{m}{n}\frac{\phi^{m-1}(\xi)}{\phi^{m(n-1)/n}(\xi)}\,\mathrm{D}_\xi\,\phi(\xi) \\ &= \frac{m}{n}\big(\phi^{m-1}\times\phi^{-m(n-1)/n}\big)(\xi)\,\mathrm{D}_\xi\,\phi(\xi) \\ &= \frac{m}{n}\phi^{(m-n)/n}(\xi)\,\mathrm{D}_\xi\,\phi(\xi) = q\phi^{q-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)\end{aligned}$$

Thus, $\mathrm{D}_\xi\big(\phi^q\big)(\xi) = q\phi^{q-1}(\xi)\,\mathrm{D}_\xi\,\phi(\xi)$ for any fractional $q \in \mathbb{Q}$. The proof of the proposition is, therefore, complete. Q.E.D.

— Corollary 4.43. *If* $\begin{array}{rl}\phi: \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi)\end{array}$ *be any differentiable function and* $q \in \mathbb{Q}$ *be any fractional, then:*

$$\mathrm{d}\big(\phi^q\big)(\xi) = q\phi^{q-1}(\xi)\,\mathrm{d}\,\phi(\xi) \tag{4.23}$$

— Corollary 4.44. *If* $\begin{array}{rl}\phi: \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi)\end{array}$ *be a function defined on an interval* $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ *by* $\phi(\xi) = \xi^q$*, where* $q \in \mathbb{Q}$ *is any fractional, then:*

– I. $\mathrm{D}_\xi\,\phi(\xi) = \mathrm{D}_\xi\big(\xi^q\big) = q\xi^{q-1}$ – II. $\mathrm{d}\,\phi(\xi) = \mathrm{d}\big(\xi^q\big) = q\xi^{q-1}\,\mathrm{d}\,\xi$

— Theorem 4.45. *If* $\begin{array}{rl}\phi,\ \psi: \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ *be any two differentiable functions, then the derivative of the product of the functions* ϕ, ψ *is the function* ϕ *times the derivative of the function* ψ *plus the function* ψ *times the derivative of the function* ϕ*:*

$$\mathrm{D}_\xi(\phi\times\psi)(\xi) = \phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi) + \psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi) \tag{4.24}$$

— Proof. Suppose $\begin{array}{rl}\phi,\ \psi: \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ be any two differentiable functions. Then,

$$\begin{aligned}\big((\phi\times\psi) + \delta(\phi\times\psi)\big)(\xi) &= (\phi+\delta\phi)(\xi)\times(\psi+\delta\psi)(\xi) \\ &= \big((\phi+\delta\phi)\times(\psi+\delta\psi)\big)(\xi) \\ &= (\phi\times\delta\psi)(\xi) + (\psi\times\delta\phi)(\xi) + (\delta\phi\times\delta\psi)(\xi)\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\delta(\phi\times\psi)}{\delta\xi}(\xi) &= (\phi+\delta\phi)(\xi)\times(\psi+\delta\psi)(\xi)\\ &= \left(\phi\times\frac{\delta\psi}{\delta\xi}\right)(\xi)+\left(\psi\times\frac{\delta\phi}{\delta\xi}\right)(\xi)+\left(\frac{\delta\phi\times\delta\psi}{\delta\xi}\right)(\xi)\end{aligned}$$

Then,

$$\begin{aligned}\mathrm{D}_\xi(\phi\times\psi)(\xi) &= \lim_{\delta\xi\to0}\frac{\delta(\phi\times\psi)}{\delta\xi}(\xi)\\ &= \lim_{\delta\xi\to0}\left(\phi\times\frac{\delta\psi}{\delta\xi}\right)(\xi)+\lim_{\delta\xi\to0}\left(\psi\times\frac{\delta\phi}{\delta\xi}\right)(\xi)+\lim_{\delta\xi\to0}\left(\frac{\delta\phi\times\delta\psi}{\delta\xi}\right)(\xi)\end{aligned}$$

The term $\lim_{\delta\xi\to0}\left(\frac{\delta\phi\times\delta\psi}{\delta\xi}\right)(\xi)$ may be written either $\left(\lim_{\delta\xi\to0}\delta\phi(\xi)\right)\times\left(\lim_{\delta\xi\to0}\frac{\delta\psi}{\delta\xi}(\xi)\right)$, or $\left(\lim_{\delta\xi\to0}\frac{\delta\phi}{\delta\xi}(\xi)\right)\times\left(\lim_{\delta\xi\to0}\delta\psi(\xi)\right)$, either of which is zero since, $\mathrm{D}_\xi\,\phi(\xi)$, $\mathrm{D}_\xi\,\psi(\xi)$ exist and $\lim_{\delta\xi\to0}\delta\phi(\xi)=\lim_{\delta\xi\to0}\delta\psi(\xi)=0$. Therefore,

$$\begin{aligned}\mathrm{D}_\xi(\phi\times\psi)(\xi) &= \phi(\xi)\times\left(\lim_{\delta\xi\to0}\frac{\delta\psi}{\delta\xi}\right)+\psi(\xi)\times\left(\lim_{\delta\xi\to0}\frac{\delta\phi}{\delta\xi}\right)\\ &= \phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi)+\psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi)\end{aligned}$$

Hence, $\mathrm{D}_\xi(\phi\times\psi)(\xi)=\phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi)+\psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi)$. The proof of the theorem is, therefore, complete. Q.E.D.

— Corollary 4.46. *If* $\begin{array}{rl}\phi,\ \psi:\ \mathscr{I}_{(\alpha,\beta)} & \longrightarrow\mathbb{R}\\ \xi & \longmapsto\phi(\xi),\ \psi(\xi)\end{array}$ *be any two differentiable functions, then the differential of the product of the functions ϕ, ψ is the function ϕ times the differential of the function ψ plus the function ψ times the differential of the function ϕ:*

$$\mathrm{d}(\phi\times\psi)(\xi)=\phi(\xi)\,\mathrm{d}\,\psi(\xi)+\psi(\xi)\,\mathrm{d}\,\phi(\xi)\tag{4.25}$$

— Illustrative Example 4.47. Let $\begin{array}{rl}\phi,\ \psi:\ \mathscr{I}_{(\alpha,\beta)} & \longrightarrow\mathbb{R}\\ \xi & \longmapsto\phi(\xi),\ \psi(\xi)\end{array}$ be two functions defined on an interval $\mathscr{I}_{(\alpha,\beta)}\subset\mathbb{R}$ by $\phi(\xi)=\gamma_1\xi+\gamma_2$ and $\psi(\xi)=\gamma_3\xi+\gamma_4$, respectively, where $(\alpha,\beta)\in\mathbb{R}^2$ and $(\gamma_1,\ldots,\gamma_4)\in\mathbb{R}^{*4}$. Find the derivative and the differential of the product function $(\phi\times\psi)(\xi)$ with respect to ξ.

Solution. Since

$$\begin{cases}\mathrm{D}_\xi\,\phi(\xi)=\mathrm{D}_\xi(\gamma_1\xi+\gamma_2)=\gamma_1\\ \mathrm{D}_\xi\,\psi(\xi)=\mathrm{D}_\xi(\gamma_3\xi+\gamma_4)=\gamma_3\end{cases}\qquad\begin{cases}\mathrm{d}\,\phi(\xi)=\mathrm{d}(\gamma_1\xi+\gamma_2)=\gamma_1\,\mathrm{d}\,\xi\\ \mathrm{d}\,\psi(\xi)=\mathrm{d}(\gamma_3\xi+\gamma_4)=\gamma_3\,\mathrm{d}\,\xi\end{cases}$$

then

$$\begin{aligned}\mathrm{D}_\xi(\phi\times\psi)(\xi) &= \phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi)+\psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi)\\ &= \big(\gamma_1\xi+\gamma_2\big)\gamma_3+\big(\gamma_3\xi+\gamma_4\big)\gamma_1=(2\gamma_1\gamma_3)\,\xi+(\gamma_1\gamma_4+\gamma_2\gamma_3)\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}(\phi\times\psi)(\xi) &= \phi(\xi)\,\mathrm{d}\,\psi(\xi)\,\mathrm{d}\,\xi+\psi(\xi)\,\mathrm{d}\,\phi(\xi)\,\mathrm{d}\,\xi\\ &= \big(\gamma_1\xi+\gamma_2\big)\gamma_3\,\mathrm{d}\,\xi+\big(\gamma_3\xi+\gamma_4\big)\gamma_1\,\mathrm{d}\,\xi\\ &= (2\gamma_1\gamma_3)\,\xi\,\mathrm{d}\,\xi+(\gamma_1\gamma_4+\gamma_2\gamma_3)\,\mathrm{d}\,\xi\end{aligned}$$

The derivative and the differential of the product function $(\phi\times\psi)(\xi)$ with respect to ξ are, therefore,

$$\begin{cases}\mathrm{D}_\xi(\phi\times\psi)(\xi)=(2\gamma_1\gamma_3)\,\xi+(\gamma_1\gamma_4+\gamma_2\gamma_3)\\ \mathrm{d}(\phi\times\psi)(\xi)=(2\gamma_1\gamma_3)\,\xi\,\mathrm{d}\,\xi+(\gamma_1\gamma_4+\gamma_2\gamma_3)\,\mathrm{d}\,\xi\end{cases}$$

respectively.

The above theorem and corollary can be extended to any number of factors, and we shall have the following two corollaries.

— Corollary 4.48. *If* $\begin{matrix}\phi_1,\ \ldots,\ \phi_n: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R}\\ & \xi & \longmapsto \phi_1(\xi),\ \ldots,\ \phi_n(\xi)\end{matrix}$ *be any* $n>1$ *differentiable functions, then the derivative of the product of the* n *functions* $\phi_1,\ \ldots,\ \phi_n$ *divided by the product is equal to the sum of the terms obtained by dividing the derivative of each of function by the function itself:*

$$\left(\frac{\mathrm{D}_\xi\left(\prod_{k\in I_n^*}\phi_k\right)}{\prod_{k\in I_n^*}\phi_k}\right)(\xi)=\sum_{k\in I_n^*}\left(\frac{\mathrm{D}_\xi\,\phi_k}{\phi_k}\right)(\xi)\tag{4.26}$$

— Corollary 4.49. *If* $\begin{matrix}\phi_1,\ \ldots,\ \phi_n: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R}\\ & \xi & \longmapsto \phi_1(\xi),\ \ldots,\ \phi_n(\xi)\end{matrix}$ *be any* $n>1$ *differentiable functions, then the differential of the product of the* n *functions* $\phi_1,\ \ldots,\ \phi_n$ *divided by the product is equal to the sum of the terms obtained by dividing the differential of each of function by the function itself:*

$$\left(\frac{\mathrm{d}\left(\prod_{k\in I_n^*}\phi_k\right)}{\prod_{k\in I_n^*}\phi_k}\right)(\xi)=\sum_{k\in I_n^*}\left(\frac{\mathrm{d}\,\phi_k}{\phi_k}\right)(\xi)\tag{4.27}$$

— Theorem 4.50. *If* $\begin{matrix}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R}\\ & \xi & \longmapsto \phi(\xi),\ \psi(\xi)\end{matrix}$ *be any two differentiable functions such that* $\psi(\xi)\neq 0$ *for each* $\xi\in\mathscr{I}_{(\alpha,\beta)}$*, then the derivative of the quotient of the functions* $\phi,\ \psi$ *is the denominator* ψ *times the derivative of the* *numerator ϕ minus the numerator ϕ times the derivative of the denominator ψ all divided by the square of the denominator function ψ:*

$$\mathrm{D}_\xi \left(\frac{\phi}{\psi}\right)(\xi) = \frac{\psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi) - \phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi)}{\left(\psi(\xi)\right)^2} \tag{4.28}$$

— Proof. Let $\begin{array}{rl} \phi,\ \psi : & \mathscr{I}_{(\alpha,\beta)} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \phi(\xi),\ \psi(\xi) \end{array}$ be any two differentiable functions such that $\psi(\xi) \neq 0$ for each $\xi \in \mathscr{I}_{(\alpha,\beta)}$. Proceed in the usual way to assign an increment $\delta\xi$ to ξ. Then,

$$\left(\frac{\phi}{\psi} + \delta\left(\frac{\phi}{\psi}\right)\right)(\xi) = \frac{(\phi+\delta\phi)(\xi)}{(\psi+\delta\psi)(\xi)}$$

Consequently,

$$\delta\left(\frac{\phi}{\psi}\right)(\xi) = \frac{(\phi+\delta\phi)(\xi)}{(\psi+\delta\psi)(\xi)} - \left(\frac{\phi}{\psi}\right)(\xi) = \frac{\psi(\xi)\times\delta\phi(\xi) - \phi(\xi)\times\delta\psi(\xi)}{\psi(\xi)\times(\psi+\delta\psi)(\xi)}$$

Then,

$$\frac{\delta(\phi/\psi)}{\delta\xi}(\xi) = \frac{\psi(\xi)\times\dfrac{\delta\phi}{\delta\xi}(\xi) - \phi(\xi)\times\dfrac{\delta\psi}{\delta\xi}(\xi)}{\psi(\xi)\times(\psi+\delta\psi)(\xi)}$$

Taking the limit of both sides as $\delta\xi \to 0$ and remembering that $\mathrm{D}_\xi\,\phi(\xi)$, $\mathrm{D}_\xi\,\psi(\xi)$ exist, and that $\lim\limits_{\delta\xi\to 0}\delta\psi(\xi) = 0$, it follows that

$$\begin{aligned}
\mathrm{D}_\xi\left(\frac{\phi}{\psi}\right)(\xi) &= \lim_{\delta\xi\to 0}\frac{\delta(\phi/\psi)}{\delta\xi}(\xi) \\
&= \lim_{\delta\xi\to 0}\frac{\psi(\xi)\times\dfrac{\delta\phi}{\delta\xi}(\xi) - \phi(\xi)\times\dfrac{\delta\psi}{\delta\xi}(\xi)}{\psi(\xi)\times(\psi+\delta\psi)(\xi)} \\
&= \frac{\psi(\xi)\times\left(\lim\limits_{\delta\xi\to 0}\dfrac{\delta\phi}{\delta\xi}(\xi)\right) - \phi(\xi)\times\left(\lim\limits_{\delta\xi\to 0}\dfrac{\delta\psi}{\delta\xi}(\xi)\right)}{\psi(\xi)\times\left(\psi(\xi) + \lim\limits_{\delta\xi\to 0}\delta\psi(\xi)\right)} \\
&= \frac{\psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi) - \phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi)}{\left(\psi(\xi)\right)^2}
\end{aligned}$$

Hence, $\mathrm{D}_\xi\left(\dfrac{\phi}{\psi}\right)(\xi) = \dfrac{\psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi) - \phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi)}{\left(\psi(\xi)\right)^2}$. The proof of the theorem is, therefore, complete. Q.e.d.

— COROLLARY 4.51. *If* $\begin{array}{rl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ *be any two differentiable functions such that* $\psi(\xi) \neq 0$ *for each* $\xi \in \mathscr{I}_{(\alpha,\beta)}$*, then the differential of the quotient of the functions* ϕ, ψ *is the denominator* ψ *times the differential of the numerator* ϕ *minus the numerator* ϕ *times the differential of the denominator* ψ *all divided by the square of the denominator function* ψ*:*

$$\mathrm{d}\left(\frac{\phi}{\psi}\right)(\xi) = \frac{\psi(\xi)\,\mathrm{d}\,\phi(\xi) - \phi(\xi)\,\mathrm{d}\,\psi(\xi)}{\left(\psi(\xi)\right)^2} \tag{4.29}$$

— ILLUSTRATIVE EXAMPLE 4.52. Let $\begin{array}{rl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} \longrightarrow \mathbb{R} \\ & \xi \longmapsto \phi(\xi),\ \psi(\xi)\end{array}$ be two functions defined on an interval $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ by $\phi(\xi) = \gamma_1\xi^2 + \gamma_2$ and $\psi(\xi) = \gamma_3\xi^2 + \gamma_4$, respectively, where $(\alpha,\beta) \in \mathbb{R}^2$ and $(\gamma_1,\ldots,\gamma_4) \in \mathbb{R}^{*4}$. Find the derivative and the differential of the quotient function $\left(\frac{\phi}{\psi}\right)(\xi)$ with respect to ξ.

SOLUTION. Since

$$\begin{cases}\mathrm{D}_\xi\,\phi(\xi) = \mathrm{D}_\xi\left(\gamma_1\xi^2+\gamma_2\right) = 2\gamma_1\xi \\ \mathrm{D}_\xi\,\psi(\xi) = \mathrm{D}_\xi\left(\gamma_3\xi^2+\gamma_4\right) = 2\gamma_3\xi\end{cases} \qquad \begin{cases}\mathrm{d}\,\phi(\xi) = \mathrm{d}\left(\gamma_1\xi^2+\gamma_2\right) = 2\gamma_1\xi\,\mathrm{d}\,\xi \\ \mathrm{d}\,\psi(\xi) = \mathrm{d}\left(\gamma_3\xi^2+\gamma_4\right) = 2\gamma_3\xi\,\mathrm{d}\,\xi\end{cases}$$

then

$$\begin{aligned}\mathrm{D}_\xi\left(\frac{\phi}{\psi}\right)(\xi) &= \frac{\psi(\xi)\,\mathrm{D}_\xi\,\phi(\xi) - \phi(\xi)\,\mathrm{D}_\xi\,\psi(\xi)}{\left(\psi(\xi)\right)^2} \\ &= \frac{\left(\gamma_3\xi^2+\gamma_4\right)(2\gamma_1\xi) - \left(\gamma_1\xi^2+\gamma_2\right)(2\gamma_3\xi)}{\left(\gamma_3\xi^2+\gamma_4\right)^2} \\ &= \frac{2\left(\gamma_1\gamma_4-\gamma_2\gamma_3\right)\xi}{\left(\gamma_3\xi^2+\gamma_4\right)^2}\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\left(\frac{\phi}{\psi}\right)(\xi) &= \frac{\psi(\xi)\,\mathrm{d}\,\phi(\xi) - \phi(\xi)\,\mathrm{d}\,\psi(\xi)}{\left(\psi(\xi)\right)^2} \\ &= \frac{\left(\gamma_3\xi^2+\gamma_4\right)(2\gamma_1\xi\,\mathrm{d}\,\xi) - \left(\gamma_1\xi^2+\gamma_2\right)(2\gamma_3\xi\,\mathrm{d}\,\xi)}{\left(\gamma_3\xi^2+\gamma_4\right)^2} \\ &= \frac{2\left(\gamma_1\gamma_4-\gamma_2\gamma_3\right)\xi}{\left(\gamma_3\xi^2+\gamma_4\right)^2}\,\mathrm{d}\,\xi\end{aligned}$$

The derivative and the differential of the quotient function $\left(\dfrac{\phi}{\psi}\right)(\xi)$ with respect to ξ are, therefore,

$$\mathrm{D}_\xi \left(\frac{\phi}{\psi}\right)(\xi) = \frac{2\left(\gamma_1\gamma_4 - \gamma_2\gamma_3\right)\xi}{\left(\gamma_3\xi^2 + \gamma_4\right)^2} \qquad \mathrm{d}\left(\frac{\phi}{\psi}\right)(\xi) = \frac{2\left(\gamma_1\gamma_4 - \gamma_2\gamma_3\right)\xi}{\left(\gamma_3\xi^2 + \gamma_4\right)^2}\,\mathrm{d}\,\xi$$

respectively.

— Corollary 4.53. *Let* $\begin{array}{rcl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\,,\ \psi(\xi)\end{array}$ *be any two differentiable functions such that* $\psi(\xi) \neq 0$ *for each* $\xi \in \mathscr{I}_{(\alpha,\beta)}$, *and let* $\kappa \in \mathbb{R}$ *is any arbitrary constant. Then:*

– *I.* $\forall \xi \in \mathscr{I}_{(\alpha,\beta)}: \ \phi(\xi) = \kappa \implies \mathrm{D}_\xi \left(\dfrac{\phi}{\psi}\right)(\xi) = -\dfrac{\kappa}{\left(\psi(\xi)\right)^2}\,\mathrm{D}_\xi\,\psi(\xi)$

– *II.* $\forall \xi \in \mathscr{I}_{(\alpha,\beta)}: \ \psi(\xi) = \kappa \implies \mathrm{D}_\xi \left(\dfrac{\phi}{\psi}\right)(\xi) = \dfrac{1}{\kappa}\,\mathrm{D}_\xi\,\phi(\xi)$

— Corollary 4.54. *Let* $\begin{array}{rcl}\phi,\ \psi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\,,\ \psi(\xi)\end{array}$ *be any two differentiable functions such that* $\psi(\xi) \neq 0$ *for each* $\xi \in \mathscr{I}_{(\alpha,\beta)}$, *and let* $\kappa \in \mathbb{R}$ *is any arbitrary constant. Then:*

– *I.* $\forall \xi \in \mathscr{I}_{(\alpha,\beta)}: \ \phi(\xi) = \kappa \implies \mathrm{d}\left(\dfrac{\phi}{\psi}\right)(\xi) = -\dfrac{\kappa}{\left(\psi(\xi)\right)^2}\,\mathrm{d}\,\psi(\xi)$

– *II.* $\forall \xi \in \mathscr{I}_{(\alpha,\beta)}: \ \psi(\xi) = \kappa \implies \mathrm{d}\left(\dfrac{\phi}{\psi}\right)(\xi) = \dfrac{1}{\kappa}\,\mathrm{d}\,\phi(\xi)$

— Theorem 4.55. *If* $\begin{array}{rcl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ *be a differentiable function on* $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ *and* $\begin{array}{rcl}\psi: & \mathscr{I}_{(\phi(\alpha),\phi(\beta))} & \longrightarrow \mathbb{R} \\ & \eta & \longmapsto \psi(\eta)\end{array}$ *be a differentiable function on* $\mathscr{I}_{(\phi(\alpha),\phi(\beta))}$, *then* $\begin{array}{rcl}\psi\circ\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \psi\circ\phi(\xi)\end{array}$ *is a differentiable function on* $\mathscr{I}_{(\alpha,\beta)}$*:*

$$\mathrm{D}_\xi\left(\psi\circ\phi\right)(\xi) = \mathrm{D}_\eta\,\psi(\eta)\,\mathrm{D}_\xi\,\phi(\xi) \tag{4.30}$$

— Proof. Let $\begin{array}{rrl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ be differentiable on $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ and let $\begin{array}{rrl}\psi: & \mathscr{I}_{(\phi(\alpha),\phi(\beta))} & \longrightarrow \mathbb{R} \\ & \eta & \longmapsto \psi(\eta)\end{array}$ be differentiable on $\mathscr{I}_{(\phi(\alpha),\phi(\beta))}$. Then,

$$\begin{aligned}
\mathrm{D}_\xi(\psi\circ\phi)(\xi) &= \lim_{h\to 0}\frac{(\psi\circ\phi)(\xi+h)-(\psi\circ\phi)(\xi)}{h} \\
&= \lim_{h\to 0}\frac{\psi\big(\eta(\xi+h)\big)-\psi\big(\eta(\xi)\big)}{h} \\
&= \lim_{h\to 0}\frac{\psi\big(\eta(\xi)+\delta\eta(\xi)\big)-\psi\big(\eta(\xi)\big)}{\eta(\xi+h)-\eta(\xi)}\times\frac{\eta(\xi+h)-\eta(\xi)}{h} \\
&= \left(\lim_{h\to 0}\frac{\delta\psi\big(\eta(\xi)\big)}{\delta\eta(\xi)}\right)\times\left(\lim_{h\to 0}\frac{\eta(\xi+h)-\eta(\xi)}{h}\right) = \mathrm{D}_\eta\psi(\eta)\,\mathrm{D}_\xi\phi(\xi)
\end{aligned}$$

Hence, $\mathrm{D}_\xi(\psi\circ\phi)(\xi) = \mathrm{D}_\eta\psi(\eta)\,\mathrm{D}_\xi\phi(\xi)$, and $\psi\circ\phi(\xi)$ is a differentiable function on $\mathscr{I}_{(\alpha,\beta)}$. The proof of the theorem is, therefore, complete. Q.e.d.

— Corollary 4.56. *If the differential of* $\begin{array}{rrl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ *on* $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ *exists and the differential of* $\begin{array}{rrl}\psi: & \mathscr{I}_{(\phi(\alpha),\phi(\beta))} & \longrightarrow \mathbb{R} \\ & \eta & \longmapsto \psi(\eta)\end{array}$ *on* $\mathscr{I}_{(\phi(\alpha),\phi(\beta))}$ *exists, then the differential of* $\begin{array}{rrl}\psi\circ\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \psi\circ\phi(\xi)\end{array}$ *on* $\mathscr{I}_{(\alpha,\beta)}$ *exists:*

$$\mathrm{d}(\psi\circ\phi)(\xi) = \frac{\mathrm{d}\psi}{\mathrm{d}\eta}(\eta)\,\mathrm{d}\phi(\xi) \tag{4.31}$$

— Illustrative Example 4.57. Let $\begin{array}{rrl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow \mathbb{R} \\ & \xi & \longmapsto \phi(\xi)\end{array}$ be a function defined on $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ by $\phi(\xi) = \gamma_1\xi^2 + \gamma_2$, and let $\begin{array}{rrl}\psi: & \mathscr{I}_{(\phi(\alpha),\phi(\beta))} & \longrightarrow \mathbb{R} \\ & \eta & \longmapsto \psi(\eta)\end{array}$ be another function defined on $\mathscr{I}_{(\phi(\alpha),\phi(\beta))}$ by $\psi(\eta) = \sqrt{\eta}$, where $(\alpha,\beta)\in\mathbb{R}^2$ and $(\gamma_1,\gamma_2)\in\mathbb{R}^{*2}$. Find the derivative and the differential of the composite function $(\psi\circ\phi)(\xi)$ with respect to ξ.

Solution. Clearly, $\eta = \phi(\xi) = \gamma_1\xi^2+\gamma_2$, implying $\psi(\eta) = \sqrt{\eta} = \sqrt{\gamma_1\xi^2+\gamma_2} = (\psi\circ\phi)(\xi)$. Since

$$\begin{cases}\mathrm{D}_\xi\phi(\xi) = \mathrm{D}_\xi\big(\gamma_1\xi^2+\gamma_2\big) = 2\gamma_1\xi \\ \mathrm{D}_\eta\psi(\eta) = \mathrm{D}_\eta\big(\sqrt{\eta}\big) = \dfrac{1}{2\sqrt{\eta}}\end{cases} \qquad \begin{cases}\mathrm{d}\phi(\xi) = \mathrm{d}(\gamma_1\xi^2+\gamma_2) = 2\gamma_1\xi\,\mathrm{d}\xi \\ \mathrm{d}\psi(\eta) = \mathrm{d}\big(\sqrt{\eta}\big) = \dfrac{1}{2\sqrt{\eta}}\,\mathrm{d}\eta\end{cases}$$

then

$$\begin{aligned}\mathrm{D}_\xi\left(\psi\circ\phi\right)(\xi) &= \mathrm{D}_\eta\,\psi(\eta)\,\mathrm{D}_\xi\,\phi(\xi)\\ &= \frac{1}{2\sqrt{\eta}}\times(2\gamma_1\xi) = \frac{\gamma_1\xi}{\sqrt{\gamma_{\gamma_1\xi}\xi^2+\gamma_2}}\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\left(\psi\circ\phi\right)(\xi) &= \frac{\mathrm{d}\,\psi}{\mathrm{d}\,\eta}(\eta)\,\mathrm{d}\,\phi(\xi)\\ &= \frac{\mathrm{d}}{\mathrm{d}\,\eta}\big(\sqrt{\eta}\big)\times\mathrm{d}\big(\gamma_1\xi^2+\gamma_2\big)\\ &= \frac{1}{2\sqrt{\eta}}\times(2\gamma_1\xi\,\mathrm{d}\,\xi) = \frac{\gamma_1\xi}{\sqrt{\gamma_{\gamma_1\xi}\xi^2+\gamma_2}}\,\mathrm{d}\,\xi\end{aligned}$$

The derivative and the differential of the composite function $(\psi\circ\phi)(\xi)$ with respect to ξ are, therefore,

$$\mathrm{D}_\xi\left(\psi\circ\phi\right)(\xi) = \frac{\gamma_1\xi}{\sqrt{\gamma_{\gamma_1\xi}\xi^2+\gamma_2}} \qquad \mathrm{d}\left(\psi\circ\phi\right)(\xi) = \frac{\gamma_1\xi}{\sqrt{\gamma_{\gamma_1\xi}\xi^2+\gamma_2}}\,\mathrm{d}\,\xi$$

respectively.

Let us now prove a theorem that permits finding the derivative of a function $\eta = \phi(\xi)$ with respect to ξ if we know the derivative of the inverse functions $\xi = \phi^{-1}(\eta)$ with respect to η.

— THEOREM 4.58. *If for the function* $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ *on* $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ *there exists an inverse function* $\begin{array}{rccl}\varphi: & \mathscr{I}_{(\phi(\alpha),\phi(\beta))} & \longrightarrow & \mathscr{I}_{(\alpha,\beta)}\\ & \eta & \longmapsto & \varphi(\eta)\end{array}$ *which at the point under consideration* $\eta \in \mathscr{I}_{(\phi(\alpha),\phi(\beta))} \subset \mathbb{R}$ *has a nonzero derivative,* $\mathrm{D}_\eta\,\varphi(\eta) \neq 0$*, then at the corresponding point* $\xi \in \mathscr{I}_{(\alpha,\beta)}$ *the function* $\phi(\xi)$ *has a derivative* $\mathrm{D}_\xi\,\phi(\xi)$ *equal to* $\dfrac{1}{\mathrm{D}_\eta\,\varphi(\eta)}$*:*

$$\mathrm{D}_\xi\,\phi(\xi) = \frac{1}{\mathrm{D}_\eta\,\varphi(\eta)} \tag{4.32}$$

— PROOF. Let for the function $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R}\\ & \xi & \longmapsto & \phi(\xi)\end{array}$ on $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ there exists an inverse function $\begin{array}{rccl}\varphi: & \mathscr{I}_{(\phi(\alpha),\phi(\beta))} & \longrightarrow & \mathscr{I}_{(\alpha,\beta)}\\ & \eta & \longmapsto & \varphi(\eta)\end{array}$ which at the point under consideration $\eta \in \mathscr{I}_{(\phi(\alpha),\phi(\beta))} \subset \mathbb{R}$ has a nonzero derivative, $\mathrm{D}_\eta\,\varphi(\eta) \neq 0$. Set

$\eta = \phi(\xi)$ and $\xi = \varphi(\eta)$. Then, differentiating, with respect to ξ, both sides of the equation $\eta = \phi(\xi)$ gives

$$1 = \mathrm{D}_\xi \xi = \mathrm{D}_\xi \varphi(\eta) = \mathrm{D}_\eta \varphi(\eta)\, \mathrm{D}_\xi \eta(\xi) = \mathrm{D}_\eta \varphi(\eta)\, \mathrm{D}_\xi \phi(\xi)$$

Thus, $\mathrm{D}_\xi \phi(\xi) = \dfrac{1}{\mathrm{D}_\eta \varphi(\eta)}$. The proof of the theorem is, therefore, complete.

Q.E.D.

Thus, the derivative of one of two reciprocal functions $\phi(\xi)$, $\varphi(\eta)$ is equal to unity divided by the derivative of the second function for corresponding values of ξ and η.

— Corollary 4.59. *If for the function* $\begin{array}{rccl}\phi: & \mathscr{I}_{(\alpha,\beta)} & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \phi(\xi)\end{array}$ *on* $\mathscr{I}_{(\alpha,\beta)} \subset \mathbb{R}$ *there exists an inverse function* $\begin{array}{rccl}\varphi: & \mathscr{I}_{(\phi(\alpha),\phi(\beta))} & \longrightarrow & \mathscr{I}_{(\alpha,\beta)} \\ & \eta & \longmapsto & \varphi(\eta)\end{array}$ *which at the point under consideration* $\eta \in \mathscr{I}_{(\phi(\alpha),\phi(\beta))} \subset \mathbb{R}$ *has a nonzero differential,* $\mathrm{d}\varphi(\eta) \neq 0$*, then at the corresponding point* $\xi \in \mathscr{I}_{(\alpha,\beta)}$ *the function* $\phi(\xi)$ *has a differential* $\mathrm{d}\phi(\xi)$ *equal to* $\dfrac{1}{\mathrm{d}\varphi(\eta)}$*:*

$$\mathrm{d}\phi(\xi) = \frac{\mathrm{d}\xi}{\mathrm{D}_\eta \varphi(\eta)} \tag{4.33}$$

— § 4.4. Concluding Remarks

In *Lesson* № 4, the definition and the basic properties of the *differential operators* $\mathrm{d}(\cdot)$, $\mathrm{D}_\xi(\cdot)$ were introduced, followed by a presentation of the *principles of differentiation*. In Lesson № 5, which immediately follows, we shall examined some fundamental techniques of obtaining the differential coefficients of various important non-elementary functions, and the presentation of *Lesson* № 4 ends here.

— § 4.5. Selected Problems

The following problems form an integral part of Assignment 2.1. on the differential operators $\mathrm{d}(\cdot)$, $\mathrm{D}_{\xi}(\cdot)$ and the principles of differentiation. Many of these problems are routine in nature. Others are more demanding. A few provide examples that are considered challenging, interesting, and instructive.

— Learning Outcomes. After doing Assignment 2.1, the student will be able to:

– I. understand how ... Differential Operators $\mathrm{d}(\cdot)$, $\mathrm{D}_{\xi}(\cdot)$ and the Principles of Differentiation,

– II. perform ... Differential Operators $\mathrm{d}(\cdot)$, $\mathrm{D}_{\xi}(\cdot)$ and the Principles of Differentiation,

– III. explain ... Differential Operators $\mathrm{d}(\cdot)$, $\mathrm{D}_{\xi}(\cdot)$ and the Principles of Differentiation,

– IV. use ... Differential Operators $\mathrm{d}(\cdot)$, $\mathrm{D}_{\xi}(\cdot)$ and the Principles of Differentiation

— Problem 4.1 (*Differentiation*). For each $k \in I_4^*$, find the average rate of change $\dfrac{\delta\phi_k}{\delta\xi}(\xi)$ and exact rate of change $\lim\limits_{\xi\to 0}\dfrac{\delta\phi_k}{\delta\xi}(\xi)$ of the function $\phi_k(\xi)$ defined as follows:

– I. $\phi_1(\xi) = 20\xi$ – II. $\phi_2(\xi) = \xi^3$

– III. $\phi_3(\xi) = \dfrac{1}{\xi}$ – IV. $\phi_4(\xi) = \sqrt{\xi}$

— Problem 4.2 (*Differentiation*). For each $k \in I_8^*$, find the average rate of change $\dfrac{\delta\phi_k}{\delta\xi}(\xi)$ and exact rate of change $\lim\limits_{\xi\to 0}\dfrac{\delta\phi_k}{\delta\xi}(\xi)$ of the function $\phi_k(\xi)$ defined as follows:

– I. $\phi_1(\xi) = 3\xi - \xi^2$ – II. $\phi_2(\xi) = 4 - 2\xi - \xi^2$

– III. $\phi_3(\xi) = \dfrac{1}{\xi} - 2\xi - \xi^2$ – IV. $\phi_4(\xi) = \dfrac{\xi}{\xi - 1}$

– V. $\phi_5(\xi) = \xi^3 - \dfrac{1}{\xi}$ – VI. $\phi_6(\xi) = 2\xi^2 - \dfrac{3}{\xi}$

– VII. $\phi_7(\xi) = \dfrac{2}{\sqrt{\xi - 3}}$ – VIII. $\phi_8(\xi) = \sqrt{\dfrac{\xi + 1}{\xi}}$

— Problem 4.3 (*Differentiation*). For each $k \in I_6^*$, differentiate the function $\phi_k(\xi)$ with respect to its variable ξ, defined as follows:

– I. $\phi_1(\xi) = 3\xi + \xi^2$ — II. $\phi_2(\xi) = \dfrac{4+5\xi^2}{\xi}$

– III. $\phi_3(\xi) = -\dfrac{8}{\xi}$ — IV. $\phi_4(\xi) = \dfrac{4-\xi^4}{6}$

– V. $\phi_5(\xi) = \dfrac{3\xi-\xi^3}{2}$ — VI. $\phi_6(\xi) = \dfrac{4+2\xi^2}{\xi}$

— Problem 4.4 (*Differentiation*). For each $k \in I_4^*$, differentiate the function $\phi_k(\xi)$ with respect to its variable ξ, defined as follows:

– I. $\phi_1(\xi) = \xi^3 + \xi - \sqrt{\xi}$ — II. $\phi_2(\xi) = \xi^2\sqrt{\xi}$

– III. $\phi_3(\xi) = -\dfrac{\sqrt{\xi}}{\xi^4}$ — IV. $\phi_4(\xi) = \sqrt{13+\xi^2}$

— Problem 4.5 (*Differentiation*). For each $k \in I_4^*$, differentiate the function $\phi_k(\xi)$ with respect to its variable ξ, defined as follows:

– I. $\phi_1(\xi) = 3\xi^3 - \dfrac{3}{\xi^3} + \dfrac{\xi^3}{3}$ — II. $\phi_2(\xi) = (3\xi-1)^2(\xi-1)^3$

– III. $\phi_3(\xi) = \left(\xi^4+4\right)^2\left(4-\xi\right)^3$ — IV. $\phi_4(\xi) = \dfrac{\left(5-\xi^2\right)}{5+\xi^2}$

— Problem 4.6 (*Differentiation*). For each $k \in I_6^*$, differentiate the function $\phi_k(\xi)$ with respect to its variable ξ, defined as follows:

– I. $\phi_5(\xi) = \sqrt{1+\xi} + \sqrt{1-\xi}$ — II. $\phi_6(\xi) = \left(\xi^4+8\right)^{3/2} - \left(8-\xi\right)^{-3/2}$

– III. $\phi_7(\xi) = (\xi+1)\sqrt{\xi^2-1}$ — IV. $\phi_8(\xi) = \dfrac{\left(\xi^3-2\right)^4}{\xi^2+2}$

– V. $\phi_9(\xi) = \dfrac{\xi}{1-\sqrt{1-\xi^2}}$ — VI. $\phi_{10}(\xi) = \left(\dfrac{3-5\xi^2}{3+5\xi^2}\right)^{1/4}$

— Problem 4.7 (*Differentiation*). For each $k \in I_4^*$, differentiate the function $\phi_k(\xi)$ with respect to its variable ξ, defined as follows:

– I. $\phi_1(\xi) = (1+\xi)\sqrt{1-\xi}$ — II. $\phi_2(\xi) = \dfrac{\xi}{\xi+\sqrt{\xi^2+1}}$

– III. $\phi_3(\xi) = \sqrt{\dfrac{1-\sqrt{\xi}}{1+\sqrt{\xi}}}$ — IV. $\phi_4(\xi) = \dfrac{\sqrt{1+\xi^2}+\sqrt{1-\xi^2}}{\sqrt{1+\xi^2}-\sqrt{1-\xi^2}}$

— PROBLEM 4.8 (*Differentiation*). For each $k \in I_4^*$, differentiate the inverse of the function $\phi_k(\xi)$ with respect to ξ, defined as follows:

– I. $\phi_1(\xi) = \dfrac{5\xi + 7}{7\xi + 9}$

– II. $\phi_2(\xi) = \left(7 - \sqrt{\xi + 3}\right)^4 + 4$

– III. $\phi_3(\xi) = \dfrac{8\xi - 3}{9 - 2\xi} + 2$

– IV. $\phi_4(\xi) = \sqrt{(3\xi + 7)^2 - 9}$

— PROBLEM 4.9 (*Differentiation*). For each $k \in I_4^*$, differentiate the inverse of the function $\phi_k(\xi)$ with respect to ξ, defined as follows:

– I. $\phi_1(\xi) =$

– II. $\phi_2(\xi) =$

– III. $\phi_3(\xi) =$

– IV. $\phi_4(\xi) =$

— PROBLEM 4.10 (*Differentiation*). For each $k \in I_4^*$, differentiate the inverse of the function $\phi_k(\xi)$ with respect to ξ, defined as follows:

– I. $\phi_1(\xi) =$

– II. $\phi_2(\xi) =$

– III. $\phi_3(\xi) =$

– IV. $\phi_4(\xi) =$