

Probability and Statistical Techniques BCNS 1206C

Unit 1: Basics of Probability

1 Random Events and Probability Concepts

1.1 Chance-Based Experiments

Probability theory begins with the notion of a random experiment an action or process that leads to uncertain outcomes. This framework, called a *probability space*, includes:

- the complete set of possible results,
- a defined collection of events,
- a method to assign likelihoods to these events.

1.2 Sample Space and Events

The totality of all outcomes is known as the *sample space* Ω . Each individual outcome is denoted by ω . Any subset of Ω is considered an event. For two events A and B :

- $A \cup B$ represents the event that either A or B (or both) occur,
- $A \cap B$ represents the event that both A and B occur,
- $A^c = \Omega \setminus A$ represents the event that A does not occur.

1.3 Event Collections

Definition 1. A collection \mathcal{F} of subsets of Ω is called an event field if:

$$\begin{aligned} \mathcal{F} &\neq \emptyset, \\ A \in \mathcal{F} &\Rightarrow A^c \in \mathcal{F}, \\ A_1, A_2, \dots \in \mathcal{F} &\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}. \end{aligned}$$

Exercise 1. Prove that if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Exercise 2. Let $A \setminus B$ denote the elements in A but not in B . Show that if $A, B \in \mathcal{F}$, then $A \setminus B \in \mathcal{F}$.

Exercise 3. Define the symmetric difference $A \Delta B$ as the set of elements in either A or B but not both. Show that $A \Delta B \in \mathcal{F}$.

2 Probability Measures

Given a sample space Ω and event field \mathcal{F} , a probability function \mathbb{P} assigns values to events such that:

- $0 \leq \mathbb{P}(A) \leq 1$ for all $A \in \mathcal{F}$,
- $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\emptyset) = 0$,
- For disjoint events A and B , $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Definition 2. A function $\mathbb{P} : \mathcal{F} \rightarrow \mathbb{R}$ is a probability measure if:

1. $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{F}$,
2. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\emptyset) = 0$,

3. For mutually exclusive events A_1, A_2, \dots , we have:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 3. A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where:

- Ω is the sample space,
- \mathcal{F} is the event field,
- \mathbb{P} is the probability measure.

Exercise 4. Show that $\mathbb{P}(A) + \mathbb{P}(A^c) = 1$ for any $A \in \mathcal{F}$.

Exercise 5. Prove that $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$.

3 Conditional Probability

Definition 4. If $\mathbb{P}(B) > 0$, the conditional probability of A given B is:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Theorem 1. If $B \in \mathcal{F}$ and $\mathbb{P}(B) > 0$, then defining $\mathbb{Q}(A) = \mathbb{P}(A|B)$ gives a valid probability measure on (Ω, \mathcal{F}) .

Exercise 6. Prove the above theorem.

Exercise 7. Show that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)\mathbb{P}(C)$.

Exercise 8. Prove that $\mathbb{P}(B|A) = \mathbb{P}(A|B) \cdot \frac{\mathbb{P}(B)}{\mathbb{P}(A)}$.

4 Independence of Events

Definition 5. Events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Definition 6. A collection $\{A_i\}_{i \in I}$ is independent if for every finite subset $J \subseteq I$:

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i).$$

Exercise 9. Let $\Omega = \{1, 2, 3, 4\}$ with equal probabilities. Show that $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{1, 4\}$ are pairwise independent but not jointly independent.

5 Law of Total Probability

Theorem 2. If $\{B_i\}$ is a partition of Ω with $\mathbb{P}(B_i) > 0$, then for any $A \in \mathcal{F}$:

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A|B_i)\mathbb{P}(B_i).$$

Exercise 10. Given: $\mathbb{P}(\text{Rain}) = \frac{2}{5}$, $\mathbb{P}(\text{Snow}) = \frac{3}{5}$, $\mathbb{P}(\text{Late}|\text{Rain}) = \frac{1}{5}$, $\mathbb{P}(\text{Late}|\text{Snow}) = \frac{3}{5}$. Find $\mathbb{P}(\text{Late})$.

6 Bayes' Theorem

Theorem 3. Let $\{B_i\}$ be a partition of Ω with $\mathbb{P}(B_i) > 0$. Then for any A with $\mathbb{P}(A) > 0$:

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_i \mathbb{P}(A|B_i)}.$$

7 Practice Problems

Problem 1. Two urns are available: Urn I holds 4 white and 3 black balls, while Urn II contains 3 white and 7 black balls. One urn is chosen at random, and a single ball is drawn. What is the chance that the selected ball is black? If the ball drawn is white, what is the probability that it came from Urn I?

Problem 2. A standard six-sided die is rolled twice. Define event B as the first roll resulting in a number less than or equal to 3, and event C as the total of both rolls being 6. Determine the probabilities of B and C , and compute the conditional probabilities $\mathbb{P}(C|B)$ and $\mathbb{P}(B|C)$.

Problem 3. Consider two urns: Urn I has 3 white and 4 black balls, and Urn II has 2 white and 6 black balls.

(a) A ball is randomly drawn from Urn I and placed into Urn II. Then, a ball is drawn from Urn II. What is the probability that this second ball is black?

(b) An urn is selected at random (each with equal likelihood), and a ball is drawn. If the ball is black, what is the probability that it came from Urn I?

Problem 4. Three urns are labeled A , B , and C . Urn A contains 3 red and 5 white marbles, Urn B has 2 red and 1 white marble, and Urn C holds 2 red and 3 white marbles. An urn is chosen at random, and a marble is drawn. If the marble is red, what is the probability that it came from Urn A ?

Problem 5. Three individuals each take one shot at a target. Their respective probabilities of hitting the target are $1/6$, $1/4$, and $1/3$. Calculate the probability that exactly one person hits the target. If only one hit is recorded, what is the probability that it was made by the first shooter?

Problem 6. A factory uses three machines A , B , and C to produce items. Machine A accounts for 50% of production, B for 30%, and C for 20%. Their respective defect rates are 3%, 4%, and 5%. What is the overall probability that a randomly selected item is defective? If a defective item is found, what is the probability it was produced by machine A ?

Problem 7. Two fair dice are rolled. Let A be the event that the first die shows an odd number, B the event that the second die shows an even number, and C the event that both dice show numbers of the same parity (either both odd or both even). Demonstrate that A , B , and C are pairwise independent but not mutually independent.

Problem 8. Prove that if events A and B are independent, then their complements $\Omega \setminus A$ and $\Omega \setminus B$ are also independent.

Problem 9. Let A represent the event that a family has children of both genders, and B the event that the family has at most one boy. Show that A and B are independent if the family has exactly three children.

Problem 10. A bag contains 6 red, 4 blue, and 5 green marbles. One marble is drawn at random. What is the probability that it is not green? If the marble drawn is red, what is the probability that it came from a bag where red marbles make up more than 40% of the total?

Problem 11. A fair coin is tossed four times. Let event A be "at least three heads occur" and event B be "the first toss is tails." Compute $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(A|B)$.

Problem 12. A company has two departments: Department X and Department Y . Department X accounts for 70% of the employees and has a 10% absentee rate. Department Y accounts for 30% of the employees and has a 5% absentee rate. What is the probability that a randomly selected employee is absent? If an employee is found to be absent, what is the probability they belong to Department X ?

Problem 13. Two cards are drawn at random from a standard deck of 52 cards without replacement. Let A be the event that both cards are red, and B be the event that at least one card is a heart. Are A and B independent?

Problem 14. A box contains 3 defective and 7 non-defective light bulbs. Two bulbs are selected at random without replacement. Let A be the event that the first bulb is defective, and B be the event that the second bulb is defective. Find $\mathbb{P}(A)$ and $\mathbb{P}(B|A)$. Are A and B independent?